A Choiceless Answer to a Question of Woodin

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In the Workshop on Set Theory held at the National University of Singapore on July 3 – July 7, 2023, Moti Gitik presented a lecture titled "On negation of the Singular Cardinals Hypothesis with GCH below". In this lecture, he discussed the following question from the 1980s due to Woodin, as well as approaches to its solution and why it is so difficult to solve:

Question: Assuming there is no inner model of ZFC with a strong cardinal, is it possible to have a model M of ZFC such that $M \vDash$ "$2^{\aleph_\omega} > \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$", together with the existence of an inner model $N^* \subseteq M$ of ZFC such that for the γ, δ so that $\gamma = (\aleph_\omega)^M$ and $\delta = (\aleph_{\omega+3})^M$, $N^* \vDash$ "γ is measurable and $2^\gamma \geq \delta$"?

Although a positive resolution of Woodin's question when both the models M and N^* satisfy AC has yet to be found, it is possible to answer his question affirmatively if the requirement that the model M satisfy AC be dropped. Specifically, in this talk, we will prove the following theorem.

Theorem 1 *For a fixed successor ordinal $\alpha = \beta+1$ where β is a successor ordinal, the theories*

(1) "ZFC + $\exists\kappa[o(\kappa) = \kappa^{+\alpha}]$"

and

(2) "ZF + $\neg AC_\omega$ + GCH holds below $\aleph_\omega$ + There is an injection $f : \aleph_{\omega+\alpha} \to \wp(\aleph_\omega)$ + There is an inner model of ZFC in which for $\gamma = \aleph_\omega$ and $\delta = \aleph_{\omega+\alpha}$, γ is measurable and $2^\gamma = \delta$"

are equiconsistent.

The hypotheses employed in Theorem 1 come from the work of Gitik. Also, the value $\alpha = 3$ provides a specific choiceless answer to Woodin's question.

When we say SCH fails in a choiceless context, we will mean an injective failure of SCH. In particular, an *injective failure of SCH at* κ means that κ is a singular limit cardinal, GCH holds below κ in the same sense as when AC is true (i.e., for every (well-ordered cardinal) $\lambda < \kappa$, there is a bijection between λ^+ and $\wp(\lambda)$), and for some $\lambda \geq \kappa^{++}$, there is an injection from λ into $\wp(\kappa)$). The above theorem obtains injective failures of SCH at $\aleph_\omega$, along with the existence of the desired inner model answering Woodin's question.

Let's also observe explicitly that the above theorem allows for counterexamples to a well-known, foundational theorem which is true assuming AC. Specifically, when $\alpha > \omega_4$, Theorem 1 provides choiceless counterexamples to Shelah's theorem that when $\aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} < \aleph_{\omega_4}$. This is of course in sharp contrast to the situation in ZFC.

The above theorem also provides an example of the phenomenon that on occasion, when the Axiom of Choice is removed from consideration, a technically challenging question or problem becomes more tractable. One may, however, end up with models satisfying conclusions that are impossible in ZFC. I will discuss this in more detail later.

A Brief Overview of the Symmetric Inner Model to be Constructed

The models to be constructed witnessing the choiceless answers to Woodin's question are *symmetric inner models of ZF*. As such, we briefly describe the general method we will use for symmetrically collapsing a singular cardinal κ which is a limit of an ω-sequence of inaccessible cardinals down to $\aleph_\omega$. In particular, our construction will result in a choiceless, symmetric inner model of a generic extension $V[G]$.

The general idea behind the construction is to start with an ω sequence of inaccessible cardinals in the ground model V, collapse them to become the $\aleph_n$ s, and then take as our witnessing model N the least model of ZF extending V which contains every finite initial segment of the sequence of collapses. N will be a proper submodel of $V[G]$, since it won't contain the ω sequence of collapses.

Getting specific, let $V \vDash$ "ZFC + $\langle \kappa_i \mid i < \omega \rangle$ is an increasing sequence of inaccessible cardinals whose limit is κ". Assume $\kappa_0 = \omega$. For $i < \omega$, let $\mathbb{P}_i = \mathrm{Coll}(\kappa_i, {<}\kappa_{i+1})$. Note that for $\delta < \delta'$ such that δ is regular and δ' is inaccessible, $\mathrm{Coll}(\delta, {<}\delta')$ is the Lévy collapse of all cardinals in the open interval (δ, δ') down to δ. We then define $\mathbb{P} = \prod_{i<\omega} \mathbb{P}_i$ with full support.

Let G be $\mathbb{P}$-generic over V, and for $i < \omega$, let G_i be the projection of G onto $\mathbb{P}_i$. For $j < \omega$, let $\mathbb{Q}_j = \prod_{i \le j} \mathbb{P}_i$ and $H_j = \prod_{i \le j} G_i$. It is the case, by the properties of the Lévy collapse and the Product Lemma, that H_j is $\mathbb{Q}_j$-generic over V. Our symmetric inner model $N \subseteq V[G]$ can now be intuitively described as the least model of ZF extending V which contains, for every $j < \omega$, the set H_j.

In order to define N more formally, we let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v}) \iff v \in V$), and symbols $\dot{H}_j$ for every $j < \omega$. N is then defined as follows.

$N_0 = \emptyset$.

$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$ if λ is a limit ordinal.

$N_{\alpha+1} = \{x \subseteq N_\alpha \mid x$ is definable over the model $\langle N_\alpha, \in, c\rangle_{c \in N_\alpha}$ via a term $\tau \in \mathcal{L}_1$ of rank $\leq \alpha\}$.

$N = \bigcup_{\alpha \in \mathrm{Ord}^V} N_\alpha$.

We will have that $N \vDash$ "$\kappa = \aleph_\omega$". Even though it won't necessarily be true that $V \vDash$ "GCH holds below κ", it always will be the case that $N \vDash$ "GCH holds below κ". Further, if $x \in N$ is a set of ordinals, then $x \in V[H_j]$ for some $j < \omega$. Hence, since $\mathbb{Q}_j \in V_\kappa$, which means that $\mathbb{Q}_j$ is well-orderable and has cardinality less than κ, the cardinal and cofinality structure in N at and above κ is the same as in V. In addition, by its construction, $N \vDash \neg \mathrm{AC}_\omega$.

The Proof of Theorem 1

To prove Theorem 1, for the duration of its proof, suppose $\alpha = \beta + 1$ where β is a successor ordinal. To show that the consistency of the theory (2) implies the consistency of the theory (1), we note that under this assumption, there is an inner model of ZFC containing a measurable cardinal γ such that $2^\gamma \geq \gamma^{+\alpha}$. By work of Gitik, this immediately implies the existence of a model of ZFC containing a measurable cardinal κ such that $o(\kappa) = \kappa^{+\alpha}$, i.e., the consistency of the theory (1).

To outline the forcing argument to be used in showing that the consistency of the theory (1) implies the consistency of the theory (2), we will start with a sufficiently large measurable cardinal κ, force to blow up κ's power set while still preserving its measurability to create a ZFC model V^{**}, add a Prikry sequence r to κ over V^{**}, and then build the choiceless inner model N just described using the Prikry sequence r. The models $V^{**} \subseteq N$ provide our choiceless answer to Woodin's question.

Specifically, suppose $V^* \vDash$ "ZFC + $\exists\kappa[o(\kappa) = \kappa^{+\alpha}]$". Without loss of generality, by passing to the appropriate inner model if necessary, we may assume that $V^* \vDash$ GCH. Therefore, by work of Gitik, we may generically extend V^* to a model V^{**} of ZFC such that $V^{**} \vDash$ "κ is measurable + $2^\kappa = \kappa^{+\alpha}$". We then do Prikry forcing over V^{**}, to obtain a model $V = V^{**}[r]$ of ZFC such that $V \vDash$ "$2^\kappa = \kappa^{+\alpha}$ + $\langle \kappa_i \mid i < \omega \rangle$ is an increasing sequence of inaccessible cardinals whose limit is κ such that $\kappa_0 = \omega$". Let $\mathbb{P}$ be the partial ordering mentioned previously, defined using the sequence $\langle \kappa_i \mid i < \omega \rangle$, and let G be $\mathbb{P}$-generic over $V = V^{**}[r]$.

We now build the choiceless inner model $N \subseteq V[G]$ as described earlier. By its construction, V^{**}, which is such that $V^{**} \subseteq V^{**}[r] = V \subseteq N \subseteq V^{**}[r][G] = V[G]$, is an inner model of N satisfying ZFC in which $\kappa = (\aleph_\omega)^N$ is measurable. As was mentioned before, the cardinal and cofinality structure at and above κ is the same in both V and N, and $N \models$ "GCH holds below κ". In addition, since V is a Prikry extension of V^{**}, the cardinals in V and V^{**} are the same. It consequently follows that $(\aleph_{\omega+\alpha})^N = (\kappa^{+\alpha})^V = (\kappa^{+\alpha})^{V^{**}}$, $N \models$ "There is an injection $f : \aleph_{\omega+\alpha} \to \wp(\aleph_\omega)$", and in V^{**}, $2^\kappa \geq (\aleph_{\omega+\alpha})^N$ (more precisely in V^{**}, $2^\kappa = (\aleph_{\omega+\alpha})^N$). Taking $M = N$ and $N^* = V^{**}$, this shows that the consistency of theory (1) implies the consistency of theory (2). This provides our choiceless answer to Woodin's question (when $\alpha = 3$), and completes the proof of Theorem 1.

$\square$

Some General Remarks

Here are some general observations and remarks:

1. It is possible to obtains analogues of Theorem 1 for singular strong limit cardinals κ of uncountable cofinality, e.g., $\aleph_{\omega_1}$, $\aleph_{\omega_2}$, etc. In the witnessing models, GCH will hold below κ, and there will be an injection from some cardinal $\lambda > \kappa^{++}$ into $\wp(\kappa)$. This contradicts Silver's ZFC theorem that GCH cannot first fail at a singular strong limit cardinal of uncountable cofinality.

2. The assumption "$\alpha = \beta+1$ where β is a successor ordinal" comes from Gitik's remark in his 1993 *APAL* paper "On measurable cardinals violating the continuum hypothesis" that it is possible to construct a model of ZFC satisfying "$2^\kappa = \kappa^{+\alpha} + \kappa$ is measurable" from a measurable cardinal κ such that $o(\kappa) = \kappa^{+\alpha}$ and α meets this requirement. This is used in the forcing portion of Theorem 1, i.e., in the proof that (1) $\Longrightarrow$ (2). For the inner model portion of the proof of Theorem 1, i.e., the proof of (2) $\Longrightarrow$ (1), which as previously noted is also due to Gitik, this assumption on α doesn't seem to be necessary.

3. As mentioned earlier, there is an interesting phenomenon that when AC is removed from consideration, a technically challenging question or problem becomes more tractable. One example of this is the question of whether the theory "ZFC + $\aleph_\omega$ is Rowbottom" is consistent, the last remaining open question from Silver's 1966 UC Berkeley doctoral dissertation (published in 1971 in *Annals Math. Logic*). This is an extremely challenging problem. Without AC, however, it follows from work of Everett Bull (unpublished by him) in his 1976 MIT doctoral dissertation and Peter Koepke (published in 2006 in *Arch. Math. Logic*) that the theories "ZF + $\neg\mathrm{AC}_\omega$ + $\aleph_\omega$ is Rowbottom" and "ZFC + There is a measurable cardinal" are equiconsistent. In particular, Bull added a Prikry sequence to a measurable cardinal and constructed the choiceless inner model discussed earlier to obtain the consistency of "ZF + $\neg\mathrm{AC}_\omega$ + $\aleph_\omega$ is Rowbottom". Koepke used a core model argument to establish the equiconsistency.

Another example of this phenomenon is the question of whether successor cardinals can satisfy the tree property. Much work has been done, and continues to be done, on this question, individually and in groups, by Cummings, Foreman, Hayut, Magidor, Mitchell, Neeman, Shelah, and Sinapova. To the best of my knowledge, there are currently limits on the longest sequence of consecutive successor cardinals starting with $\aleph_2$ that can satisfy the tree property known to be possible. Without AC, however, it is possible, starting with a proper class of supercompact cardinals, to force and construct a choiceless model of ZF + DC in which every successor cardinal is regular and satisfies the tree property. In this model, $\aleph_1$ also satisfies the tree property, something known to be impossible assuming AC.

We end by reiterating Woodin's question, and ask if a positive answer to it can be found assuming that the model M satisfies AC.

Thank you all very much for your attention!