# UA and the Number of Normal Measures over $\aleph_{\omega+1}$

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I will begin by reviewing some preliminary material. The *Mitchell ordering on normal measures over a measurable cardinal* $\kappa$, introduced by Mitchell in his 1974 JSL paper "Sets Constructible from Sequences of Ultrafilters", is defined by $\mathcal{U}_0 \lhd \mathcal{U}_1$ for normal measures $\mathcal{U}_0$, $\mathcal{U}_1$ over $\kappa$ iff $\mathcal{U}_0 \in V^\kappa/\mathcal{U}_1$. It is known that the Mitchell ordering is well-founded. The *Mitchell order of the normal measure* $\mathcal{U}$, $o(\mathcal{U})$, is the rank of $\mathcal{U}$ in $\lhd$. The *Mitchell order of* $\kappa$*,* $o(\kappa)$, is the height of $\lhd$. Assuming GCH, the maximal value of $o(\kappa)$ is $\kappa^{++}$.

The *Ultrapower Axiom UA*, introduced by Goldberg and Woodin, has been extensively studied by Goldberg, including in his 2019 doctoral dissertation supervised by Woodin, his 2018 JML paper "The Linearity of the Mitchell Order", and additional papers submitted for publication. It says the following:

Suppose $V \models$ ZFC and $\mathcal{U}_0$, $\mathcal{U}_1 \in V$ are countably complete ultrafilters over $x_0 \in V$, $x_1 \in V$ respectively, with $j_{\mathcal{U}_0} : V \to M_{\mathcal{U}_0}$ and $j_{\mathcal{U}_1} : V \to M_{\mathcal{U}_1}$ the associated elementary embeddings. Then there exist $\mathcal{W}_0 \in M_{\mathcal{U}_0}$ an $M_{\mathcal{U}_0}$-countably complete ultrafilter over $y_0 \in M_{\mathcal{U}_0}$ and $\mathcal{W}_1 \in M_{\mathcal{U}_1}$ an $M_{\mathcal{U}_1}$-countably complete ultrafilter over $y_1 \in M_{\mathcal{U}_1}$ such that:

1. For $j_{\mathcal{W}_0} : M_{\mathcal{U}_0} \to M_{\mathcal{W}_0}$ and $j_{\mathcal{W}_1} : M_{\mathcal{U}_1} \to M_{\mathcal{W}_1}$ the associated elementary embeddings, $M_{\mathcal{W}_0} = M_{\mathcal{W}_1} = M$.

2. $j_{\mathcal{W}_0} \circ j_{\mathcal{U}_0} = j_{\mathcal{W}_1} \circ j_{\mathcal{U}_1}$.

The following commutative diagram illustrates UA pictorially:

$$
\begin{array}{ccc}
V & \xrightarrow{\;j_{\mathcal{U}_0}\;} & M_{\mathcal{U}_0} \\
\Big\downarrow{\scriptstyle j_{\mathcal{U}_1}} & & \Big\downarrow{\scriptstyle j_{\mathcal{W}_0}} \\
M_{\mathcal{U}_1} & \xrightarrow{\;j_{\mathcal{W}_1}\;} & M
\end{array}
$$

UA is known to be true in the usual inner models constructed at lower levels of the large cardinal hierarchy. As Goldberg stated in his 2018 JML paper, UA also holds in inner models constructed by Woodin and Neeman-Steel for finite levels of supercompactness using iteration hypotheses. UA has therefore been studied in more generalized contexts, including models containing (fully) supercompact cardinals.

Goldberg has shown that UA has many important, interesting, and beautiful consequences. One of these is the Kimchi-Magidor property, i.e., that in any model with supercompact cardinals, the supercompact and strongly compact cardinals coincide, except at measurable limits. For the purposes of this lecture, we will be interested in Theorem 2.5 of Goldberg's 2018 JML paper, which tells us that UA implies the Mitchell ordering over any measurable cardinal must be linear. As Goldberg has noted, this means that UA implies any supercompact cardinal must carry only one normal measure concentrating on non-measurable cardinals.

Using Theorem 2.5, UA allows us to infer the following easy proposition, which will be key to the upcoming discussion.

**Proposition 1** *Assume UA. Let* $\gamma = |\delta|$*. If* $\lambda$ *is a measurable cardinal such that* $o(\lambda) = \delta$*, then the number of normal measures* $\lambda$ *carries is* $\gamma$*.*

**Proof:** Because UA holds, the Mitchell ordering over any measurable cardinal must be linear (and in fact, must be a well-ordering, since the Mitchell ordering is well-founded). Let $\mathbb{U}$ be the set of normal measures over $\lambda$. The function $f : \delta \to \mathbb{U}$ given by

$f(\alpha) =$ The unique normal measure over $\lambda$ of Mitchell order $\alpha$

therefore is well-defined and is a bijection between $\delta$ and $\mathbb{U}$. Thus, the number of normal measures $\lambda$ carries is $\gamma$. $\square$

We turn now to the main topic of this lecture, namely investigating the number of normal measures consistently possible at the successor of a singular cardinal. We will focus on the specific example of $\aleph_{\omega+1}$, although the methods discussed will be applicable in the context of other successors of singular cardinals (regardless of cofinality). In particular, it is known that assuming AD + DC, $\aleph_{\omega+1}$ is a measurable cardinal and carries exactly three normal measures.

This raises the general

Question: How many normal measures can $\aleph_{\omega+1}$ carry when $\aleph_{\omega+1}$ is a measurable cardinal?

By an earlier theorem, starting from a model of ZFC containing cardinals $\kappa < \lambda$ in which $\kappa$ is supercompact and $\lambda$ is measurable, one can force and construct a choiceless inner model of ZF in which $\aleph_{\omega+1}$ is measurable and carries exactly $\gamma$ normal measures, where $\gamma \geq \aleph_{\omega+2}$ is an arbitrary regular cardinal. UA is not needed in the proof. However, UA now allows us to construct analogous models in which $\gamma \leq \aleph_{\omega+1}$. Specifically, we have the following:

**Theorem 1** *Suppose* $V \vDash$ *"ZFC + UA +* $\kappa < \lambda$ *are such that* $\kappa$ *is supercompact,* $\lambda$ *is measurable, and* $o(\lambda) = \delta$ *for* $\delta \le \lambda^{++}$*". There is then a partial ordering* $\mathbb{P} \in V$ *and a symmetric submodel* $N \subseteq V^{\mathbb{P}}$ *such that* $N \vDash$ *"ZF +* $\aleph_{\omega+1}$ *is measurable and carries exactly* $\gamma = |\delta|$ *normal measures".*

Thus, by a judicious choice of $\delta$, we may assume that in $N$, $\aleph_{\omega+1}$ carries exactly $\gamma = 1,\ 2,$ $75,\ \omega,\ \aleph_{57},\ \aleph_\omega,\ \aleph_{\omega+1},$ etc. normal measures. Also, depending on the exact definition of $\mathbb{P}$, $N$ may be constructed so that AC either fails completely or the maximal amount of AC consistent with the measurability of $\aleph_{\omega+1}$, $\mathrm{DC}_{\aleph_\omega}$, holds. In addition, core model theory tells us that strong hypotheses beyond the existence of one measurable cardinal (i.e., at least a Woodin cardinal) must be used in order to establish Theorem 1.

Turning now to a discussion of the proof of Theorem 1, let $V$ be as in its hypotheses. To simplify the presentation, the forcing conditions to be given will create a final model $N$ in which $\mathrm{DC}_\kappa$ holds, $\kappa$ is a singular cardinal having cofinality $\omega$, and $\kappa^+ = \lambda$ is measurable and carries exactly $|\delta|$ normal measures.

(Remark: To prove Theorem 1 as stated, one either uses a more complicated version of the forcing conditions (due to Magidor) to symmetrically collapse $\kappa$ to $\aleph_\omega$ and $\lambda$ to $\aleph_{\omega+1}$ (if one wishes to have $\mathrm{DC}_{\aleph_\omega}$ be true), or (following a suggestion of Kanamori) one forces over $N$ to symmetrically collapse $\kappa$ to be $\aleph_\omega$ and $\lambda$ to be $\aleph_{\omega+1}$ (if one doesn't mind a final model in which AC fails completely).)

To define the forcing conditions $\mathbb{P}$ employed in the construction of the witnessing model $N$, let $\mathcal{U}$ be a normal measure over $P_\kappa(\lambda)$ satisfying the Menas partition property, which is an analogue for supercompactness of the partition property for normal measures over a measurable cardinal given by Rowbottom's theorem. $\mathbb{P}$ is then defined as the set of all conditions of the form $\pi = \langle p_1, \ldots, p_n, A \rangle$ such that:

1. $n \in \omega$.
2. $p_i \in P_\kappa(\lambda)$ for $i = 1, \ldots, n$.
3. $p_1 \subsetneq \cdots \subsetneq p_n$, where $p_i \subsetneq p_j$ means $p_i \subseteq p_j$ and $\text{otp}(p_i) < p_j \cap \kappa$.
4. $A \in \mathcal{U}$.
5. For each $p \in A$, $p_n \subsetneq p$.

For any condition $\pi$ as just described, call $\langle p_1, \ldots, p_n \rangle$ *the $p$-part of* $\pi$.

The ordering on $\mathbb{P}$ is $\pi_2 = \langle q_1, \ldots, q_m, B \rangle$ extends $\pi_1 = \langle p_1, \ldots, p_n, A \rangle$ iff:

1. $n \leq m$.

2. $q_i = p_i$ for $i = 1, \ldots, n$.

3. $q_i \in A$ for $i = n + 1, \ldots, m$.

4. $B \subseteq A$.

Let $G$ be $V$-generic over $\mathbb{P}$. Let $r = \langle p_i \mid i \in \omega \rangle$ be the $\omega$ sequence generated by $G$, i.e., $p_i \in r$ iff $\exists \pi \in G[p_i \in \mathrm{p-part}(\pi)]$. In $V[G]$, $\kappa$ is a cardinal, and by a density argument, $\mathrm{cf}(\gamma) = \omega$ for all $V$-regular cardinals $\gamma \in [\kappa, \lambda]$. By induction, all $V$-cardinals $\gamma \in (\kappa, \lambda]$ are collapsed to $\kappa$.

$V[G]$, being a model of AC in which $\lambda$ is no longer a cardinal, is not the desired model $N$ witnessing the conclusions of Theorem 1. To define $N$, for any $\gamma \in [\kappa, \lambda)$ an inaccessible cardinal in $V$, let $r \restriction \gamma = \langle p_i \cap \gamma \mid i \in \omega \rangle$. In $V[r \restriction \gamma]$, all $V$-cardinals $\eta \in (\kappa, \gamma]$ are collapsed to $\kappa$, and all $V$-regular cardinals $\eta \in [\kappa, \gamma]$ have cofinality $\omega$. $N$ may now be described intuitively as the least model of ZF extending $V$ which contains, for each $V$-inaccessible cardinal $\gamma \in [\kappa, \lambda)$, the set $r \restriction \gamma$.

It can be shown that $N \vDash$ "ZF + $\mathrm{DC}_\kappa$ + $\kappa$ is a cardinal having cofinality $\omega$ + $\lambda = \kappa^+$ is measurable and carries normal measures". In addition, for every normal measure $\mu^*$ over $\lambda$ in $N$, there exists a normal measure $\mu$ over $\lambda$ in $V$ such that $\mu^* = \{x \subseteq \lambda \mid \exists y \in \mu[y \subseteq x]\}$. Thus, there is a 1-1 correspondence in $N$ between the normal measures over $\lambda$ in $V$ and the normal measures over $\lambda$ in $N$. Since by Proposition 1, UA implies that $\lambda$ carries precisely $|\delta|$ normal measures in $V$, this 1-1 correspondence implies $N \vDash$ "$\lambda$ carries precisely $|\delta|$ normal measures". This completes the discussion of the proof of Theorem 1.

$\square$

Some remarks:

1. As was mentioned earlier, strong hypotheses beyond the existence of one measurable cardinal are required to construct models in which the successor of a singular cardinal is measurable and carries normal measures. To see this, suppose $\kappa$ is singular and $\kappa^+$ is measurable. By doing Prikry forcing using one of the normal measures $\kappa^+$ carries, we may change the cofinality of $\kappa^+$ to $\omega$ without adding bounded subsets of $\kappa^+$. This allows us to obtain a model in which both $\kappa$ and $\kappa^+$ are singular. By a theorem of Schindler, this means there must be an inner model containing a Woodin cardinal.

2. The methods outlined are applicable to the successors of other singular cardinals. In particular, assuming UA, $\aleph_{\omega+1}$ can be replaced by $\aleph_{\omega+\omega+1}$, $\aleph_{\omega^2+1}$, $\aleph_{\aleph_{\omega^2}+1}$, or other successors of singular cardinals of cofinality $\omega$. It is not even necessary to restrict consideration to successors of singular cardinals of countable cofinality. It is also possible assuming UA to obtain analogous results for $\aleph_{\omega_1+1}$ and other successors of singular cardinals of uncountable cofinality.

3. Global results may also be obtained from UA together with suitable additional hypotheses. As an example, it is possible to construct a model of ZF in which AC fails completely, yet every successor cardinal is regular, every limit cardinal is singular, and the successor of every singular cardinal is measurable and carries, e.g., exactly one normal measure.

4. We do not need strong hypotheses to force the successor of a regular cardinal to be measurable and control the number of normal measures. In particular, if we wish to obtain results analogous to Theorem 1 for successors of regular cardinals, then UA is unnecessary, and it suffices to use only one measurable cardinal. As an example, suppose we wish to construct a model in which $\aleph_1$ is measurable and carries, e.g., exactly two normal measures. We simply do a Friedman-Magidor style forcing over $L_\mu$ to obtain a model with exactly two normal measures and then symmetrically collapse the measurable cardinal $\kappa$ to $\aleph_1$. There is nothing special about $\aleph_1$ in this construction. For instance, essentially the same proof can be used to create, starting from one measurable cardinal, a model in which $\aleph_{\omega_5+8}$ is measurable and carries exactly $\aleph_{36}$ normal measures, etc.

For each of the results described in Theorem 1 and (2) and (3) above, UA is key to the proof. Also, if the number of normal measures $\gamma$ at the successor of the singular cardinal $\kappa$ in question is to be such that $1 \leq \gamma \leq \kappa^+$, the proofs require that the measurable cardinal $\lambda$ collapsed to $\kappa^+$ be such that $o(\lambda) = \delta$ for the appropriate $\delta$. These lead to the following two questions:

1. Can UA be removed as a hypothesis?

2. Can the requirement that $o(\lambda) = \delta$ be removed and somehow be replaced by a Friedman-Magidor style argument to control the number of normal measures $\lambda$ carries?

My conjecture is that failing the existence of inner models for supercompactness hypotheses that also satisfy the appropriate fine structural properties, the answer to both of these questions is no.

Thank you all very much for your attention!