Indestructibility and the First Two Strongly Compact Cardinals

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CUNY Set Theory Seminar
April 24, 2020

I will begin by stating the following well-known result due to Richard Laver.

Theorem 1 *Suppose $V \models$ "ZFC + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ is supercompact and has its supercompactness indestructible under κ-directed closed forcing".*

In other words, Theorem 1 says that if $\mathbb{Q} \in V^{\mathbb{P}}$ is such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ-directed closed" (i.e., every directed subset of $\mathbb{Q}$ of size less than κ has a lower bound), then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is supercompact".

One may wonder if there are analogous results when κ is a non-supercompact strongly compact cardinal. In particular, are there indestructibility theorems when κ is both the least strongly compact and least measurable cardinal (a situation first shown to be possible by Magidor)?

The answer to this question is yes. Specifically, we have the following (where for the duration of the lecture, the terminology to be used is that κ has its strong compactness indestructible under a particular type of partial ordering if after forcing with that type of partial ordering, κ remains strongly compact):

Theorem 2 (A.-Gitik '98)

Suppose $V \models$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ is both the least strongly compact and least measurable cardinal and has its strong compactness indestructible under κ-directed closed forcing".

Because the partial ordering $\mathbb{P}$ used in the proof of Theorem 2 is an Easton support Prikry iteration as first developed by Gitik, iterating a suitably defined version of $\mathbb{P}$ above above a strongly compact cardinal λ will destroy λ's strong compactness. However, by iterating strategically closed forcing, Sargsyan was able to establish:

Theorem 3 (Sargsyan '09)

Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ is both the least strongly compact and least measurable cardinal and has its strong compactness indestructible under any κ-directed closed partial ordering preserving $2^\kappa = \kappa^+$".

It is now possible to combine Theorems 2 and 3 to obtain:

Theorem 4 *Suppose* $V \models$ *"ZFC + GCH +* $\kappa_1 < \kappa_2$ *are both supercompact + No cardinal* $\lambda > \kappa_2$ *is measurable". There is then a partial ordering* $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"*κ_1 *and* κ_2 *are both the first two strongly compact and first two measurable cardinals +* κ_1*'s strong compactness is indestructible under* κ_1*-directed closed forcing +* κ_2*'s strong compactness is indestructible under any* κ_2*-directed closed partial ordering preserving* $2^{\kappa_2} = \kappa_2^+$*".*

To prove Theorem 4, let V be as in the hypotheses. First force with the partial ordering used in the proof of Theorem 2 defined with respect to κ_1 and then force with the partial ordering used in the proof of Theorem 3 defined with respect to κ_2. Since the partial ordering used in the proof of Theorem 3 may be defined so as to be κ_1-directed closed, this provides us with the desired model. $\square$

Because of the nature of Sargsyan's partial ordering, the indestructibility forced for κ_2 in Theorem 3 requires that GCH at κ_2 be preserved. This led him to ask the following

Question: Is it possible to obtain a model of ZFC in which the first two strongly compact cardinals κ_1 and κ_2 are also the first two measurable cardinals and κ_2's strong compactness is indestructible under $\mathrm{Add}(\kappa_2, \kappa_2^{++})$ (where as usual, $\mathrm{Add}(\kappa_2, \delta)$ for δ an ordinal is the standard poset for adding δ many Cohen subsets of κ_2)?

The answer to a generalized version of this Question is yes and forms the basis for the remainder of my lecture. Specifically, we have the following:

Theorem 5 *Suppose $V \models$ "ZFC + GCH + $\kappa_1 < \kappa_2$ are both supercompact + No cardinal $\lambda > \kappa_2$ is measurable". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ_1 and κ_2 are both the first two strongly compact and first two measurable cardinals + κ_1's strong compactness is indestructible under κ_1-directed closed forcing + κ_2's strong compactness is indestructible under $\mathrm{Add}(\kappa_2, \delta)$ for any ordinal δ".*

To prove Theorem 5, let V be as in the hypotheses. We may now divide the proof into the following four modules.

Module 1: Force over V with the partial ordering used in the proof of Theorem 2 defined with respect to κ_1. Call the resulting model V_1. Since this partial ordering may be defined so as to have cardinality κ_1, $V_1 \models$ "κ_1 is both the least strongly compact and least measurable cardinal + GCH holds at and above κ_1 + κ_1's strong compactness is indestructible under κ_1-directed closed forcing + κ_2 is supercompact + No cardinal $\lambda > \kappa_2$ is measurable".

Module 2: Force over V_1 with the Easton support iteration of length κ_2 which adds, to each measurable cardinal in the open interval (κ_1, κ_2), a non-reflecting stationary set of ordinals of cofinality κ_1. Call the resulting model V_2. By its definition, this partial ordering is κ_1-directed closed. Further, by an argument due to Magidor, $V_2 \models$ "κ_2 is strongly compact and is the least measurable cardinal greater than κ_1". It therefore follows that $V_2 \models$ "κ_1 and κ_2 are both the first two strongly compact and first two measurable cardinals + κ_1's strong compactness is indestructible under κ_1-directed closed forcing".

Module 3: Force over V_2 using Woodin's notion of fast function forcing to add a fast function f for κ_2. (Fast function forcing is defined as $\{p \mid p : \kappa_2 \to \kappa_2$ is a function such that dom(p) consists of inaccessible cardinals in the open interval (κ_1, κ_2), $|p \restriction \lambda| < \lambda$ for every inaccessible cardinal $\lambda \leq \kappa_2$, and for every $\delta \in$ dom(p), $p''\delta \subseteq \delta\}$, ordered by inclusion.) Call the resulting model V_3. Since as just defined, fast function forcing is κ_1-directed closed, $V_3 \models$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1's strong compactness is indestructible under κ_1-directed closed forcing". Arguments due to Hamkins also show that $V_3 \models$ "κ_2 is both the second strongly compact and second measurable cardinal".

Module 4: First, recall that for $\mathcal{A}$ a collection of partial orderings, the *lottery sum* $\oplus\mathcal{A} = \{\langle \mathbb{P}, p\rangle \mid \mathbb{P} \in \mathcal{A}$ and $p \in \mathbb{P}\} \cup \{1\}$, ordered with 1 above everything and $\langle \mathbb{P}, p\rangle \le \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. (Intuitively, the generic object over the lottery sum $\oplus\mathcal{A}$ first selects a poset in $\mathcal{A}$ and then forces with it.) Keeping this in mind, force over V_3 with the Easton support iteration of length κ_2 which, if $\alpha \in (\kappa_1, \kappa_2)$ is an inaccessible cardinal such that $f''\alpha \subseteq \alpha$, forces with $\bigoplus_{\beta < f(\alpha)} \mathrm{Add}(\alpha, \beta)$. Call the resulting model V_4. Because this partial ordering is κ_1-directed closed, $V_4 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1's strong compactness is indestructible under κ_1-directed closed forcing". By arguments due to Hamkins, $V_4 \vDash$ "No cardinal in the open interval (κ_1, κ_2) is measurable". By arguments due to Usuba, $V_4 \vDash$ "κ_2 is strongly compact (and hence is the second measurable cardinal) and has its strong compactness indestructible under forcing with $\mathrm{Add}(\kappa_2, \delta)$ for any ordinal δ".

V_4 is thus the desired model in which the first two strongly compact cardinals κ_1 and κ_2 are also the first two measurable cardinals, κ_1's strong compactness is indestructible under κ_1-directed closed forcing, and κ_2's strong compactness is indestructible under forcing with $\mathrm{Add}(\kappa_2, \delta)$ for any ordinal δ. $\square$

Note that the key to the proof of Theorem 5 is the use of Usuba's arguments referred to in Module 4.

I will conclude by asking whether Theorem 5 can be improved to have κ_2's strong compactness indestructible under arbitrary κ_2-directed closed forcing. In general, can the first i strongly compact cardinals κ_i (for i some ordinal) all be the first i measurable cardinals and have their strong compactness indestructible under arbitrary κ_i-directed closed forcing?

Thank you all very much for your attention!