From Strong to Woodin cardinals. A level-by-level analysis of the Weak Vopěnka Principle

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CUNY Set Theory Seminar (Online)
1 May 2020

Vopěnka's Principle

Definition (P. Vopěnka 1960's)

Vopěnka's Principle (VP) asserts that there is no rigid proper class of graphs.

Some equivalent forms of VP:

- (Solovay-Reinhardt-Kanamori) For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- (Adámek-Rosičký) $\mathtt{Ord}$ cannot be fully embedded into $\mathtt{Gra}$. i.e., there is no sequence $\langle G_\alpha : \alpha \in \mathrm{ORD}\rangle$ of graphs such that for every $\alpha \leqslant \beta$ there exists exactly one homomorphism $G_\alpha \to G_\beta$, and no homomorphism $G_\beta \to G_\alpha$ whenever $\alpha < \beta$.

The Weak Vopěnka's Principle (WVP)

Definition (Adámek-Rosický-Trnková 1988[1])

The Weak Vopěnka's Principle asserts that $\mathtt{Ord}^{\mathtt{op}}$ cannot be fully embedded into $\mathtt{Gra}$.

I. e., there is no sequence $\langle G_\alpha : \alpha \in \mathrm{ORD} \rangle$ of graphs such that for every $\alpha \leqslant \beta$ there exists exactly one homomorphism $G_\beta \to G_\alpha$, and no homomorphism $G_\alpha \to G_\beta$ whenever $\alpha < \beta$.

The motivation for this principle was the following:

- VP holds iff every full subcategory of a locally presentable category closed under colimits is coreflective.
- WVP holds iff every full subcategory of a locally presentable category closed under limits is reflective.

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[1] J. Adámek, J. Rosický, and V. Trnková. *Are all limit-closed subcategories of locally presentable categories reflective?* In Categorical algebra and its applications (Louvain-La-Neuve, 1987), volume 1348 of Lecture Notes in Math., pages 1-18. Springer, Berlin, 1988.

VP vs. WVP

Proposition

VP implies WVP.

Using a result of R. Isbell (1960), which shows that $\mathtt{Ord}^{\mathtt{op}}$ is bounded iff there is no proper class of measurable cardinals, Adámek-Rosický (1994) show that WVP implies a proper class of measurable cardinals.

Question (Adámek-Rosický 1988)

Does WVP imply VP?

Wilson's Theorem

Theorem (T. Wilson 2019)

The following are equivalent:

1. WVP
2. ORD *is Woodin.*

"ORD is Woodin" means: For every proper class A there is some cardinal κ such that for every λ there exists a non-trivial elementary embedding $j : V \to M$, with M transitive, $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda = j(A) \cap V_\lambda$.

This shows that WVP does not imply VP, because VP is known to imply the the existence of large cardinals much stronger (consistency-wise) than "ORD is Woodin".

The WVP for definable classes

Definition

Let $n > 0$. The Σ_n-*Weak Vopěnka Principle* (Σ_n-WVP) asserts that there exists no Σ_n-definable sequence $\langle G_\alpha : \alpha \in \mathrm{ORD} \rangle$ of graphs such that for every $\alpha \leqslant \beta$ there exists exactly one homomorphism $G_\beta \to G_\alpha$, and no homomorphism $G_\alpha \to G_\beta$ whenever $\alpha < \beta$.

Π_n-WVP is defined similarly.

Strong cardinals

Recall that a cardinal κ is strong if for every ordinal λ there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit} j = \kappa$, and $V_\lambda \subseteq M$.

If κ is strong, then $\kappa \in C^{(2)}$, i.e., $V_\kappa \preceq_{\Sigma_2} V$.

More generally, we define:

Definition

A cardinal κ is Σ_n-*strong* if for every Σ_n-definable (with parameters in V_κ) class A, for every ordinal λ there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

Every strong cardinal is Σ_2-strong.
Also, if $\lambda \in C^{(n+1)}$, then a cardinal is λ-Π_n-strong iff it is λ-Σ_{n+1}-strong. So, Π_n-strong $\equiv$ Σ_{n+1}-strong.

Main Theorem

Theorem (BW[2])

The following are equivalent for $n \geqslant 2$:

1. Σ_n-WVP
2. *There exists a Σ_n-strong cardinal.*

In particular, Σ_2-WVP holds iff there exists a strong cardinal.

[2] Bagaria, J. and Wilson, T. *The Weak Vopěnka Principle for definable classes of structures*. Preprint.

The Γ—PRP principle

The first step in the proof of the Theorem is a reformulation of WVP in terms of a Product Reflection Principle:

Definition (BW)

For Γ a definability class (i.e., one of Σ_n, Π_n, some $n > 0$), the *Product Reflection Principle* Γ-PRP asserts that for every Γ-definable proper class $\mathcal{C}$ of relational structures of the same type there exists a cardinal κ that *product-reflects* $\mathcal{C}$, i.e.,

For every X *in* $\mathcal{C}$ *there is a homomorphism* $\prod(\mathcal{C} \cap V_\kappa) \to X$.

The equivalence of Γ-PRP and Γ-WVP

Theorem (BW)

Γ-PRP and Γ-WVP are equivalent, for every definability class Γ.

The proof first shows that Γ-PRP is equivalent to Γ-SWVP (where SWVP is the *Semi-Weak Vopěnka Principle*, which is the same as WVP but we don't require uniqueness of the downward arrows). Clearly Γ-SWVP implies Γ-WVP.

Then it shows that Γ-WVP implies Γ-SWVP, and therefore they are equivalent.

Main Theorem (full version)

Theorem (BW)

The following are equivalent for $n \geqslant 2$:

1. *There exists a Σ_n-strong cardinal.*
2. *There exists a Π_{n-1}-strong cardinal.*
3. Σ_n-PRP
4. Π_{n-1}-PRP
5. Σ_n-WVP
6. Π_{n-1}-WVP
7. Σ_n-SWVP
8. Π_{n-1}-SWVP

There are only two remaining non-trivial implications. Namely,

- (1)$\Rightarrow$(3): If there exists a Σ_n-strong cardinal, then Σ_n-PRP holds.
- (4)$\Rightarrow$(2): Π_{n-1}-PRP implies the existence of a Π_{n-1}-strong cardinal.

Proof of (1)⇒(3):

Let κ be Σ_n-strong and let $\mathcal{C}$ be definable, by a Σ_n formula with parameters in V_κ, proper class of structures in a fixed relational language.

Given any $\mathcal{A} \in \mathcal{C}$, let $\lambda \geqslant \kappa$ be such that $\mathcal{A} \in V_\lambda$.

Let $j : V \to M$ be an elementary embedding, with $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $j(\kappa) > \lambda$, and $\mathcal{C} \cap V_\lambda \subseteq j(\mathcal{C})$.

By elementarity, the restriction of j to $\mathcal{C} \cap V_\kappa$ yields a homomorphism

$$h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(j(\mathcal{C}) \cap V^M_{j(\kappa)}).$$

Since $\mathcal{A} \in \mathcal{C} \cap V_\lambda$, we have that $\mathcal{A} \in j(\mathcal{C})$. Moreover $\mathcal{A} \in V_\lambda \subseteq V^M_{j(\kappa)}$. Thus, letting

$$g : \prod(j(\mathcal{C}) \cap V^M_{j(\kappa)}) \to \mathcal{A}$$

be the projection map, we have that

$$g \circ h : \prod(\mathcal{C} \cap V_\kappa) \to \mathcal{A}$$

is a homomorphism, as wanted. $\square$

The hard implication is:

- (4)$\Rightarrow$(2): Π_{n-1}-PRP implies the existence of a Π_{n-1}-strong cardinal.

Idea of the Proof (for the basic case $n = 2$):

A cardinal μ is λ-strong if and only if $V_{\lambda+1} \models$ "μ is λ-strong", since the λ-strongness of μ is witnessed by a strong (μ, λ)-extender which, if it exists, belongs to $V_{\lambda+1}$.
Let $\mathcal{A}$ be the class of all structures

$$\mathcal{A}_\alpha := \langle V_{\lambda_\alpha+1}, \in, \alpha, \mu_\alpha, \lambda_\alpha, \{R^\alpha_\varphi\}_{\varphi\in\Pi_1}\rangle$$

where μ_α is the α-th element of $C^{(1)}$, λ_α is the least cardinal in $C^{(1)}$ greater than μ_α such that no cardinal $\leqslant \mu_\alpha$ is λ_α-strong, and $\{R^\alpha_\varphi\}_{\varphi\in\Pi_1}$ is the Π_1 relational diagram for $V_{\lambda_\alpha+1}$, i.e., if $\varphi(x_1, \dots, x_n)$ is a Π_1 formula in the language of $\langle V_{\lambda_\alpha+1}, \in, \alpha, \mu_\alpha, \lambda_\alpha\rangle$, then

$$R^\alpha_\varphi = \{\langle x_1, \dots, x_n\rangle : \langle V_{\lambda_\alpha+1}, \in, \alpha, \mu_\alpha, \lambda_\alpha\rangle \models \text{“}\varphi(x_1, \dots, x_n)\text{”}\}.$$

$\mathcal{A}$ is Π_1-definable without parameters. For $X \in \mathcal{A}$ if and only if $X = \langle X_0, X_1, X_2, X_3, X_4, X_5 \rangle$, where

(1) X_2 is an ordinal

(2) X_3, X_4 belong to $C^{(1)}$

(3) $X_0 = V_{X_4+1}$

(4) $X_1 = \in \restriction X_0$

(5) X_5 is the Π_1 relational diagram of $\langle X_0, X_1, X_2, X_3, X_4 \rangle$, and

(6) the following hold in $\langle X_0, X_1, X_2, X_3, X_4 \rangle$:

  0.1 V_{X_4} satisfies that X_3 is the X_2-th element of $C^{(1)}$

  0.2 $\forall \kappa \leqslant X_3(\kappa$ is not X_4-strong$)$

  0.3 $\forall \mu(\mu \in C^{(1)} \wedge X_3 < \mu < X_4 \rightarrow \exists \kappa \leqslant X_3(\kappa$ is μ-strong$))$.

If there is no strong cardinal, then $\mathcal{A}$ is a proper class.

By Π_1-PRP there exists a cardinal $\kappa \in C^{(2)}$ such that for every ordinal β there is a homomorphism

$$j_\beta : \prod(\mathcal{C} \cap V_\kappa) \to \mathcal{A}_\beta$$

Notice that, since $\kappa \in C^{(2)}$,

$$\prod(\mathcal{C} \cap V_\kappa) = \langle \prod_{\alpha<\kappa} V_{\lambda_\alpha+1}, \overline{\in}, \langle \alpha \rangle_{\alpha<\kappa}, \langle \mu_\alpha \rangle_{\alpha<\kappa}, \langle \lambda_\alpha \rangle_{\alpha<\kappa}, \{\overline{R}^\alpha_\varphi\}_{\varphi \in \Pi_1} \rangle$$

where $\overline{\in}$ is the pointwise membership relation, and $\overline{R}^\alpha_\varphi$ is the pointwise R^α_φ relation.

Now fix some β greater than λ_κ of uncountable cofinality and such that $\mu_\beta = \beta$, and let $j = j_\beta$.

Define $k : V_{\kappa+1} \to V_{\beta+1}$ by

$$k(X) = j(\langle X \cap V_{\mu_\alpha} \rangle_{\alpha<\kappa}) \,.$$

Claim

1. k *preserves the Boolean operations, as well as the* $\subseteq$ *relation.*
2. k *maps ordinals to ordinals, and is the identity on* $\omega + 1$*.*

Note that $k(\kappa) = j(\langle \mu_\alpha \rangle_{\alpha<\kappa}) = \mu_\beta = \beta$.

For each $a \in [\beta]^{<\omega}$, define E_a by

$$X \in E_a \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k(X)\,.$$

Since $k(\kappa) = \beta$ and $k(|a|) = |a|$, we also have $k([\kappa]^{|a|}) = [\beta]^{|a|}$, hence $[\kappa]^{|a|} \in E_a$. Moreover, since k preserves Boolean operations and the $\subseteq$ relation, E_a is an ultrafilter over $[\kappa]^{|a|}$. Furthermore, since $k(\omega) = \omega$, E_a is ω_1-complete.

Hence, the ultrapower $\mathrm{Ult}(V, E_a)$ is well-founded. Let

$$j_a : V \to M_a \cong \mathrm{Ult}(V, E_a)$$

be the corresponding ultrapower embedding.

Claim

$\mathcal{E} = \{E_a : a \in [\beta]^{<\omega}\}$ *is normal and coherent.*

Let $M_{\mathcal{E}}$ be the direct limit of

$$\langle\langle M_a : a \in [\beta]^{<\omega}\rangle, \langle i_{ab} : a \subseteq b\rangle\rangle$$

where the $i_{ab} : M_a \to M_b$ are the usual commuting maps.

Let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ be the corresponding limit elementary embedding.

Using the normality of $\mathcal{E}$, and the fact that β has uncountable cofinality, we can show:

Claim

$M_{\mathcal{E}}$ *is closed under* ω*-sequences, hence it is well-founded.*

Let $\pi : M_{\mathcal{E}} \to N$ be the transitive collape.

Claim

$V_\beta \subseteq N$

Let $j_N : V \to N$ be the composition map $j_N = \pi \circ j_{\mathcal{E}}$.

Claim

$j_N(\kappa) \geqslant \beta$.

Since $\beta > \kappa$, this implies that the critical point of j_N is less than or equal to κ. Thus, since $V_\beta \subseteq N$, j_N witnesses that the critical point of j_N is a β-strong cardinal. But this contradicts our choice of λ_κ. $\square$

For the general case (i.e., $n \geqslant 2$) we work with the Π_{n-1}-definable class of structures

$$\mathcal{A}_\alpha := \langle V_{\lambda'_\alpha}, \in, \alpha, \mu_\alpha, \lambda_\alpha, \lambda'_\alpha, C^{(n-1)} \cap \mu_\alpha, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1} \rangle$$

α an ordinal, where μ_α is the α-th element of $C^{(n-1)}$, λ_α is the least cardinal in $C^{(n-1)}$ greater than μ_α such that no cardinal $\leqslant \mu_\alpha$ is λ_α-Π_{n-1}-strong, λ'_α is the least ordinal in $C^{(n-1)}$ greater than λ_α, and $\{R^\alpha_\varphi\}_{\varphi \in \Pi_1}$ is the Π_1 relational diagram for $V_{\lambda'_\alpha}$.

The proof is as before, but one still needs to show that $A \cap V_\beta \subseteq j_N(A)$ to conclude that $\mathrm{crit}(j_N)$ is $\beta-\Pi_{n-1}$-strong.

For this, we use the following:

Proposition

Suppose that $n \geqslant 2$ and β is a limit point of $C^{(n-1)}$. Then the following are equivalent for any cardinal $\kappa < \beta$:

1. *κ is β-Π_{n-1}-strong, i.e., for every Π_{n-1}-definable class A there is an elementary embedding $j : V \to M$, with M transitive, $crit(j) = \kappa$, $V_\beta \subseteq M$, and $A \cap V_\beta \subseteq j(A)$.*
2. *There is an elementary embedding $j : V \to M$, with M transitive, $crit(j) = \kappa$, $V_\beta \subseteq M$, and $M \models$ "$\beta \in C^{(n-1)}$".*

The proof is then completed by showing:

Claim

$N \models$ "$\beta \in C^{(n-1)}$"

for then j_N witnesses that its critical point is a $\beta{-}\Pi_{n-1}$-strong cardinal, in contradiction to our choice of λ_κ. $\square$

Product Structural Reflection

Recall:

SR: *For every (definable) class of relational structures $\mathcal{C}$ of the same type there exists α such that α* **reflects** *$\mathcal{C}$, i.e.,*
for every $\mathcal{A}$ in $\mathcal{C}$ there exists $\mathcal{B}$ in $\mathcal{C} \cap V_\alpha$ and an (elementary) embedding from $\mathcal{B}$ into $\mathcal{A}$.

Now define:

PSR: *For every (definable) class of relational structures $\mathcal{C}$ of the same type there exists α such that α* **product-reflects** *$\mathcal{C}$, i.e.,*
for every $\mathcal{A}$ in $\mathcal{C}$ there exists a set $\mathcal{S}$ with $\mathcal{A} \in \mathcal{S}$ and an (elementary) embedding from $\prod(\mathcal{C} \cap V_\alpha) \to \prod \mathcal{S}$.

PSR: From Strong to Woodin cardinals

Complexity	SR	PSR
Σ_1	ZFC	ZFC
Π_1, Σ_2	Supercompact	Strong
Π_2, Σ_3	Extendible	Π_2-Strong
Π_3, Σ_4	$C^{(2)}$-Extendible	Π_3-Strong
$\vdots$	$\vdots$	$\vdots$
Π_n, Σ_{n+1}	$C^{(n-1)}$-Extendible	Π_n-Strong
$\vdots$	$\vdots$	$\vdots$
Π_n, all n	VP	ORD is Woodin