I-small submodels of countable models of arithmetic

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Set Theory vs Arithmetic

Set Theory	Arithmetic
Cardinal	Cut
Regular cardinal	Semiregular cut
Weakly compact cardinal	Strong cut

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- Independence results

I-small sets

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- A subset X of M is *I-small* if there exists some function $f \in M$ such that $f \restriction_I$ is a bijection from I onto X. Equivalently, X is I-small iff there exists some $a \in M$ such that:

$$(1)\ X = \{(a)_i : \ i \in I\}, \text{ and}$$
$$(2)\ (a)_i \neq (a)_j \text{ for all distinct } i, j \in I.$$

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- If $I = \mathbb{N}$, then we simply use *small* instead of $\mathbb{N}$-small.
- First appearance: Lascar 1994, *Small index property*.

Outline of the talk

- Properties of I-small subsets of $\mathcal{M}$.
- Automorphism group of a countable recursively saturated model of PA and I-small submodels.
- Initial self-embeddings of countable models of $\mathrm{I}\Sigma_1$ and I-small submodels.

Properties of I-small subsets of $\mathcal{M}$

Which subsets of M are I-small?

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> - **(Ackermann's membership relation).** There exists a Δ_0-formula $x\mathrm{E}y$ asserting that *"the x-th bit of the binary expansion of y is 1"*. a_{E} denotes the set of E-members of a in $\mathcal{M}$.
> - $\mathrm{SSy}_I(\mathcal{M}) := \{X \cap I : X \text{ is } \Sigma_1\text{-definable in } \mathcal{M}\} = \{a_{\mathrm{E}} \cap I : \ a \in M\}$.
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$A := \{i \in I : \ \mathcal{M} \models \neg\, i\mathrm{E}(a)_i\} \neq \emptyset$ is inside $\mathrm{SSy}_I(\mathcal{M})$ but not in $\mathrm{SSy}_I(\mathcal{M}_0)$.

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- By Compactness Theorem, there exists some elementary extension $\mathcal{N}$ of $\mathcal{M}$ in which $\mathcal{M}$ is small.

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Notation:

- Let $\langle \delta_r : \; r \in M \rangle$ be a *canonical enumeration* of all Δ_0-formulas within $\mathcal{M}$.
- The predicate $\mathrm{Sat}_{\Delta_0}(x)$ is the truth predicate for Δ_0-formulas in $\mathcal{M}$, which is Δ_1-definable in $\mathcal{M}$.
- For every $r \in M$, $f_r(\bar{x}) = y$ denotes the following partial Σ_1-function in $\mathcal{M}$:

$$y := \text{the least element such that } \exists z \, \mathrm{Sat}_{\Delta_0}(\delta_r(\bar{x}, y, z)).$$

- The notation $[f_r(\bar{x}) \downarrow]$ denotes the Σ_1-formula $\exists z, y \, \mathrm{Sat}_{\Delta_0}(\delta_r(\bar{x}, y, z))$, and $[f_r(\bar{x}) \downarrow]^{<w}$ stands for the formula $\exists z, y < w \, \mathrm{Sat}_{\Delta_0}(\delta_r(\bar{x}, y, z))$.
- Let $\mathcal{F}$ be the collection of all $\emptyset$-definable partial Σ_1-functions in $\mathcal{M}$.

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It is easy to see that:

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$\mathrm{K}^1(\mathcal{M}; c) = \{f_n(c) : \ n \in \mathbb{N} \text{ and } \mathcal{M} \models [f_n(c) \downarrow]\}$. Fix some nonstandard $s \in M$, and let $a \in M$ such that:

$$\mathcal{M} \models \forall r < s \begin{pmatrix} ([f_r(c) \downarrow] \to (a)_r = f_r(c)) \wedge \\ (\neg [f_r(c) \downarrow] \to (a)_r = 0) \end{pmatrix}.$$

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- $\mathrm{K}^1(\mathcal{M}; I)$ is the subset of Σ_1-definable elements of $\mathcal{M}$ with elements of I as parameter.

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$g(0) := (a)_0$, and

$g(x+1) := (a)_r$ s.t. r is the least element for which $(a)_r$ is not between elements of $\{g(z) : z \leq x\}$.

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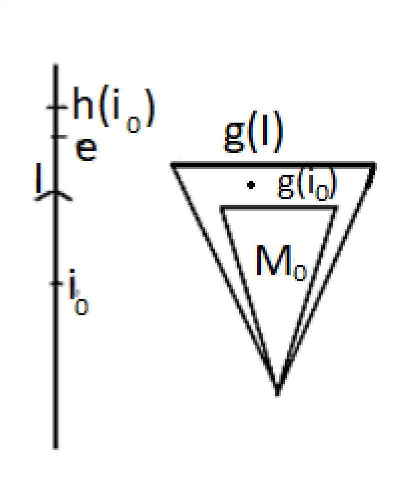

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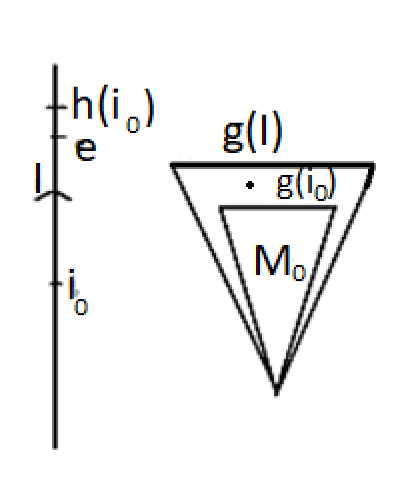

So $M_0 = \{g(i) : \ i < i_0\}$, which is a contradiction. As a result, $g \restriction_I$ is a bijection from I onto $\mathrm{K}^1(\mathcal{M}; I)$.

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(2) If $\mathcal{M}$ is a countable *recursively saturated* model of PA, then:

2-1) **(Kossak-Schmerl (1995)).** There exists some small recursively saturated elementary submodel $\mathcal{M}_0$ of $\mathcal{M}$ which has $2^{\aleph_0}$ elementary submodels.

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Let S be a *satisfaction class* for $\mathcal{M}$ such that $\mathcal{M}^* := (\mathcal{M}; S)$ is also recursively saturate. Then put $\mathcal{M}_0 := \mathrm{K}(\mathcal{M}^*)$

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$$\text{Suppose } M_0 := \{(a)_i : \ i \in I\}. \text{ Then}$$
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Suppose $M_0 := \{(a)_i : i \in I\}$. Then $X := I \cap \{\langle y, z\rangle \in M : \mathcal{M} \models (a)_y = z\}$ is inside $\mathrm{SSy}_I(\mathcal{M})$. Now, if $I \not\subseteq M_0$, then $(I; X) \models \exists x\, (\forall y\, \langle y, x\rangle \notin X)$.

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Question.

Is the strongness of I necessary in the previous statements?

I-small submodels and automorphisms of $\mathcal{M}$

Results about automorphisms of $\mathcal{M}$

Schmerl (in Kaye-Kossak-Kotlarski's 1991 paper)

Suppose $\mathcal{M}$ is a countable recursively saturated model of PA, I is a cut of $\mathcal{M}$, and $\mathcal{M}_0$ is an I-small elementary submodel of $\mathcal{M}$. Then I is strong in $\mathcal{M}$ iff there exists some automorphism j of $\mathcal{M}$ such that $M_0 = \mathrm{Fix}(j)$.

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Suppose $\mathcal{M}$ is a countable recursively saturated model of PA. Then:

I) for every small elementary submodel of $\mathcal{M}_0$ and every automorphism j of $\mathcal{M}$, $M_0 \cap \mathrm{Fix}(j)$ is small in $\mathcal{M}$.

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I) for every small elementary submodel of $\mathcal{M}_0$ and every automorphism j of $\mathcal{M}$, $M_0 \cap \mathrm{Fix}(j)$ is small in $\mathcal{M}$.

II) The following are equivalent:
   1) $\mathbb{N}$ is strong in $\mathcal{M}$.
   2) For every small $\mathcal{M}_0 \prec \mathcal{M}$ there exists some automorphism j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = M_0$.
   3) There exists some automorphism j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathcal{M}(0)$.
   4) There exists some automorphism j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \not\cong \mathcal{M}$.

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Kossak-Kotlarski (1996)

Suppose $\mathcal{M}$ is a countable recursively saturated model of PA, $\mathcal{M}_0 = \{(a)_n : n \in \mathbb{N}\}$ is a small elementary submodel of $\mathcal{M}$ and j is an automorphism $\mathcal{M}_0$. Then there exists an automorphism $\hat{j}$ of $\mathcal{M}$ which extends j iff there exists some $b \in M$ such that $j((a)_n) = (b)_n$ for all $n \in \mathbb{N}$, and the same holds for j^{-1}.

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Enayat (2006)

Suppose $\mathcal{M} \models \mathrm{PA}$ is countable, recursively saturated, and I is a strong cut of $\mathcal{M}$. Moreover, let $\mathcal{M}_0$ be an I-small elementary submodel of $\mathcal{M}$. Then there exists a group embedding Φ from $\mathrm{Aut}(\mathbb{Q}, <)$ into $\mathrm{Aut}(\mathcal{M})$ such that for every fixed point free automorphism j of $(\mathbb{Q}, <)$ it holds that $\mathrm{Fix}(\Phi(j)) = M_0$.

I-small submodels and initial self-embeddings of $\mathcal{M}$

Friedman’s Theorem

Friedman (1973)

Let $\mathcal{M}$, $\mathcal{N}$ be countable nonstandrd models of PA. The following statements are equivalent:

(1) $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N})$.

(2) There is an embedding $j : \mathcal{M} \to \mathcal{N}$ such that $j(\mathrm{M}) \subset_e \mathcal{N}$.

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There are many refinements of Friedman's Theorem in the literature. In particular, Ressayre proved a similar result for models of $\mathrm{I}\Sigma_1$. Moreover, Dimitracopoulos and Paris developed a version of Friedman's Theorem for models of $\mathrm{I}\Delta_0 + \mathrm{Exp}$.

I-small submodels as fixed point

B-Enayat (2018)

Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard and I is a cut of $\mathcal{M}$. Then the following hold:

(1) I is strong in $\mathcal{M}$ and $I \prec_{\Sigma_1} \mathcal{M}$, iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = I$.

(2) $\mathbb{N}$ is strong in $\mathcal{M}$ iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = \mathrm{K}^1(\mathcal{M})$.

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(2) $\mathbb{N}$ is strong in $\mathcal{M}$ iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = \mathrm{K}^1(\mathcal{M})$.

B (2022)

Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard, I is a cut, and $\mathcal{M}_0$ is an I-small Σ_1-elementary submodel of $\mathcal{M}$. Then I is strong in $\mathcal{M}$ iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = M_0$.

Sketch of proof of left to right:

- $\mathrm{I}^1(\mathcal{M};X) := \{x : \ x \leq a \text{ for some } a \in \mathrm{K}^1(\mathcal{M};X)\} \prec_{\Sigma_0} \mathcal{M}$;
- $\mathrm{H}^1(\mathcal{M};X) := \bigcup_{k\in\omega} \mathrm{H}^1_k(\mathcal{M};X)$, where:

$$\mathrm{H}^1_0(\mathcal{M};X) := \mathrm{I}^1(\mathcal{M};X), \text{ and}$$
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- $\mathrm{H}^1(\mathcal{M};X) \prec_{\Sigma_1} \mathcal{M}$ and $\mathrm{H}^1(\mathcal{M};X) \models \mathrm{I}\Sigma_1$.

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(i) We will construct some proper initial self-embedding α of $\mathrm{H}^1(\mathcal{M}; M_0)$ such that $\mathrm{Fix}(\alpha) = M_0$ and $\alpha(\mathrm{H}^1(\mathcal{M}; M_0)) < b$ for some $b \in \mathrm{H}^1(\mathcal{M}; M_0)$.

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(i) We will construct some proper initial self-embedding α of $\mathrm{H}^1(\mathcal{M};M_0)$ such that $\mathrm{Fix}(\alpha) = M_0$ and $\alpha(\mathrm{H}^1(\mathcal{M};M_0)) < b$ for some $b \in \mathrm{H}^1(\mathcal{M};M_0)$.

(ii) By $\mathrm{I}\Sigma_1$-version of the Friedman's Theorem, let $\beta : \mathcal{M} \hookrightarrow \mathrm{H}^1(\mathcal{M};M_0)$ be a proper initial embedding such that $M_0 \subset \mathrm{Fix}(\beta)$ and $b \in \beta(M)$.

Sketch of proof of left to right:

- $\mathrm{I}^1(\mathcal{M}; X) := \{x : \ x \leq a \text{ for some } a \in \mathrm{K}^1(\mathcal{M}; X)\} \prec_{\Sigma_0} \mathcal{M}$;
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(i) We will construct some proper initial self-embedding α of $\mathrm{H}^1(\mathcal{M}; M_0)$ such that $\mathrm{Fix}(\alpha) = M_0$ and $\alpha(\mathrm{H}^1(\mathcal{M}; M_0)) < b$ for some $b \in \mathrm{H}^1(\mathcal{M}; M_0)$.

(ii) By $\mathrm{I}\boldsymbol{\Sigma}_1$-version of the Friedman's Theorem, let $\beta : \mathcal{M} \hookrightarrow \mathrm{H}^1(\mathcal{M}; M_0)$ be a proper initial embedding such that $M_0 \subset \mathrm{Fix}(\beta)$ and $b \in \beta(M)$.

(iii) Finally, put $j := \beta^{-1}\alpha\beta$.

Construction of α:

- First by using strong Σ_1-Collection in $\mathrm{H}^1(\mathcal{M}; M_0)$, we will find some $b \in \mathrm{H}^1(\mathcal{M}; M_0)$ such that:

$$\mathcal{M} \models [f((a)_i)\downarrow] \rightarrow [f((a)_i)\downarrow]^{<b}, \text{ for all } f \in \mathcal{F} \text{ and all } i \in I.$$

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- **Back and forth:** We will build finite functions $\bar{u} \mapsto \bar{v}$ of elements of $\mathrm{H}^1(\mathcal{M}; M_0)$ such that the following properties hold:
  - $\mathrm{P}(\bar{u}, \bar{v}, i, f) \equiv [f(\bar{u}, (a)_i)\downarrow] \rightarrow [f(\bar{v}, (a)_i)\downarrow]^{<b}$, for all $f \in \mathcal{F}$ and $i \in I$,

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  - $\mathrm{Q}(\bar{u}, \bar{v}, i, f) \equiv \begin{pmatrix} [f(\bar{u}, (a)_i) \downarrow] \wedge \\ [f(\bar{v}, (a)_i) \downarrow]^{<b} \wedge \\ f(\bar{u}, (a)_i) \notin M_0 \end{pmatrix} \Rightarrow f(\bar{u}, (a)_i) \neq f(\bar{v}, (a)_i)$, for all $f \in \mathcal{F}$ and all $i \in I$.

Construction of α:

Note that $\mathrm{Q}(\bar{u}, \bar{v}, i, f)$ can be written as a Π_1-formula; to be more exact, let:

$R :=$

$$\left\{ \langle k, t \rangle \in I : \mathrm{H}^1(\mathcal{M}; M_0) \models \left(\begin{array}{c} ([f(\bar{u}, (a)_i) \downarrow] \wedge [f_t(\bar{u}, (a)_k) \downarrow]) \rightarrow \\ f(\bar{u}, (a)_i) = f_t(\bar{u}, (a)_k) \end{array} \right) \right\}.$$

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R is Π_1-definable and so coded in $\mathrm{H}^1(\mathcal{M}; M_0)$.

Construction of α:

- **Forth levels (for making domain of α to be equal to $\mathrm{H}^1(\mathcal{M}; M_0)$):** Suppose $\bar{u} \mapsto \bar{v}$ is constructed and $m \in \mathrm{H}^1(\mathcal{M}; M_0)$ is arbitrary. So w.l.o.g. we can assume that $m \leq t(\bar{u}, (a)_{i_0})$ for some $t \in \mathcal{F}$ and $i_0 \in I$.

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  So w.l.o.g. we can assume that $m \leq t(\bar{u}, (a)_{i_0})$ for some $t \in \mathcal{F}$ and $i_0 \in I$.
  For every $s \in \mathrm{H}^1(\mathcal{M}; M_0)$ be arbitrary and put:
  $p_s(y) :=$
  $$\{y \leq t(\bar{v}, (a)_{i_0})\} \cup \left\{ \forall i, i' < s \begin{pmatrix} \mathrm{P}(\bar{u}, m, \bar{v}, y, i, f) \wedge \\ \mathrm{Q}(\bar{u}, m, \bar{v}, y, i', f') \end{pmatrix} : \ f, f' \in \mathcal{F} \right\}.$$

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  Our aim is to find some $s > I$ such that the bounded Π_1-type $p_s(y)$ is finitely satisfiable.

Construction of α:

Define the following function:

$$\mathrm{G}(x) := \max\{s < b : \mathcal{M} \models \Theta(s, x, \bar{u}, m, \bar{v})\},$$

in which $\Theta(s, x, \bar{u}, m, \bar{v})$ is the following Δ_0-formula:

$$\forall r, r' < x \, \exists y \leq t(\bar{v}, (a)_{i_0}) \, \forall i, i' < s(\mathrm{P}(\bar{u}, m, \bar{v}, y, i, f_r) \wedge \mathrm{Q}(\bar{u}, m, \bar{v}, y, i', f_{r'})).$$

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Now by strongness of I there exists some $e > I$ such that $\mathrm{G}(i) > I$ iff $\mathrm{G}(i) > e$ for all $i \in I$.

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Intuitively, $\mathrm{G}(x)$ is the largest element s less than b such that $p_s(y)$ is satisfiable for elements f_r such that $r < x$.

Now by strongness of I there exists some $e > I$ such that $\mathrm{G}(i) > I$ iff $\mathrm{G}(i) > e$ for all $i \in I$. We will show that $p_e(y)$ is a finitely satisfiable type.

Construction of α:

Lemma

For every finite number of elements of $\mathcal{F}$, say $f, f_{n_1}, \ldots, f_{n_k}$, there exists some $s > I$ such that:

$$\mathcal{M} \models \exists y < t(\bar{v}, (a)_{i_0}) \forall i, i' < s \, (\mathrm{P}(\bar{u}, m, \bar{v}, y, i, f) \wedge \bigwedge_{w=1}^{k} \mathrm{Q}(\bar{u}, m, \bar{v}, y, i', f_{n_w})).$$

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First not that by $\mathrm{I}\Sigma_1$-version of Friedman's Theorem, for all $s > I$ it holds that:
$\mathrm{H}^1(\mathcal{M}; M_0) \models \exists y < t(\bar{v}, (a)_{i_0}) \, \forall i < s \, \mathrm{P}(\bar{u}, m, \bar{v}, y, i, f)$, for all $f \in \mathcal{F}$.

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So in order to prove $p_e(y)$ is finitely satisfiable, suppose $f \in \mathcal{F}$ and $f_{n_1}, ..., f_{n_k} \in \mathcal{F}$ are given.

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The standard cut

Corollary

Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ *be countable and nonstandard. T.F.A.E:*

1) $\mathbb{N}$ *is strong in* $\mathcal{M}$.
2) *For every small* $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$ *there exists some proper initial self-embedding* j *of* $\mathcal{M}$ *such that* $\mathrm{Fix}(j) = M_0$.
3) *There exists some proper initial self-embedding* j *of* $\mathcal{M}$ *such that* $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$.

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Corollary

Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ be countable and nonstandard. T.F.A.E:

1) *$\mathbb{N}$ is strong in $\mathcal{M}$.*
2) *For every small $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$ there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = M_0$.*
3) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$.*

If $\mathcal{M} \models \mathrm{PA}$ is recursively saturated, then the above statements are equivalent to the following:

4) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$ and it is isomorphic to no proper initial segments of $\mathcal{M}$.*

The standard cut

Proof.

(3) $\Rightarrow$ (1) : There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$.

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(3) $\Rightarrow$ (1) : There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$. First by strong Σ_1-collection axiom in $\mathcal{M}$, there exists some $b \in M \backslash \mathrm{I}^1(\mathcal{M})$.

Proof.

(3) $\Rightarrow$ (1) : There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$. First by strong Σ_1-collection axiom in $\mathcal{M}$, there exists some $b \in M \backslash \mathrm{I}^1(\mathcal{M})$. Now, suppose $\mathbb{N}$ is not strong in $\mathcal{M}$.

Proof.

(3) $\Rightarrow$ (1) : There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$. First by strong Σ_1-collection axiom in $\mathcal{M}$, there exists some $b \in M \backslash \mathrm{I}^1(\mathcal{M})$. Now, suppose $\mathbb{N}$ is not strong in $\mathcal{M}$.

> **Lemma.** Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ in which $\mathbb{N}$ is not a strong cut, and j is a self-embedding of $\mathcal{M}$, then for every element $b \in M$ there exists an element $c \in \mathrm{Fix}(j)$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}; b) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; c)$.

Proof.

(3) $\Rightarrow$ (1) : There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$. First by strong Σ_1-collection axiom in $\mathcal{M}$, there exists some $b \in M \backslash \mathrm{I}^1(\mathcal{M})$. Now, suppose $\mathbb{N}$ is not strong in $\mathcal{M}$.

> **Lemma.** Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ in which $\mathbb{N}$ is not a strong cut, and j is a self-embedding of $\mathcal{M}$, then for every element $b \in M$ there exists an element $c \in \mathrm{Fix}(j)$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}; b) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; c)$.

As a result, there exists some $c \in \mathrm{Fix}(j)$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}; b) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; c)$. Which is a contradiction. □

Proof.

(2) $\Rightarrow$ (4): Let $\mathcal{M}_0$ be the small recursively saturated elementary submodel of $\mathcal{M}$ we talked about in the previous slides.

The standard cut

Proof.

(2) $\Rightarrow$ (4): Let $\mathcal{M}_0$ be the small recursively saturated elementary submodel of $\mathcal{M}$ we talked about in the previous slides. So by (2) there exists some proper initial self-embedding j of $\mathcal{M}$ s.t. $\mathrm{Fix}(j) = M_0$.

The standard cut

Proof.

(2) $\Rightarrow$ (4): Let $\mathcal{M}_0$ be the small recursively saturated elementary submodel of $\mathcal{M}$ we talked about in the previous slides. So by (2) there exists some proper initial self-embedding j of $\mathcal{M}$ s.t. $\mathrm{Fix}(j) = M_0$. Since M_0 is small, $\mathrm{SSy}(\mathcal{M}_0) \neq \mathrm{SSy}(\mathcal{M})$, so it cannot be isomorphic to any initial segments of $\mathcal{M}$.

Proof.

(2) $\Rightarrow$ (4): Let $\mathcal{M}_0$ be the small recursively saturated elementary submodel of $\mathcal{M}$ we talked about in the previous slides. So by (2) there exists some proper initial self-embedding j of $\mathcal{M}$ s.t. $\mathrm{Fix}(j) = M_0$. Since M_0 is small, $\mathrm{SSy}(\mathcal{M}_0) \neq \mathrm{SSy}(\mathcal{M})$, so it cannot be isomorphic to any initial segments of $\mathcal{M}$.
(4) $\Rightarrow$ (1): There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$ and it is isomorphic to no proper initial segments of $\mathcal{M}$.

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Proof.

(2) $\Rightarrow$ (4): Let $\mathcal{M}_0$ be the small recursively saturated elementary submodel of $\mathcal{M}$ we talked about in the previous slides. So by (2) there exists some proper initial self-embedding j of $\mathcal{M}$ s.t. $\mathrm{Fix}(j) = M_0$. Since M_0 is small, $\mathrm{SSy}(\mathcal{M}_0) \neq \mathrm{SSy}(\mathcal{M})$, so it cannot be isomorphic to any initial segments of $\mathcal{M}$.
(4) $\Rightarrow$ (1): There exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$ and it is isomorphic to no proper initial segments of $\mathcal{M}$. If $\mathbb{N}$ is not strong, by the previous Lemma for every $a \in M$, there exists some $b \in \mathrm{Fix}(j)$ such that $\mathbb{N} \cap a_{\mathrm{E}} = \mathbb{N} \cap b_{\mathrm{E}}$. As a result, $\mathrm{SSy}(\mathrm{Fix}(j)) = \mathrm{SSy}(\mathcal{M})$.

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I-small submodels and extendability of initial self-embeddings of $\mathcal{M}$

Theorem (B 2022)

Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard, I is a strong cut of $\mathcal{M}$, $\mathcal{M}_0$ is an I-small Σ_1-elementary submodel of $\mathcal{M}$ such that $M_0 := \{(a)_i : i \in I\}$, and j is an initial self-embedding of $\mathcal{M}_0$ such that $j(I) \subseteq_e \mathcal{M}$. Then the following are equivalent:

(1) *j extends to some proper initial self-embedding of $\mathcal{M}$.*

(2)
- *There exists some $b \in M$ such that $\mathcal{M} \models j((a)_i) = (b)_{j(i)}$ for all $i \in I$, and*
- *for every subset A of M_0 it holds that:*

$$A \in \mathrm{SSy}_I(\mathcal{M}) \text{ iff } j(A) \in \mathrm{SSy}_{j(I)}(\mathcal{M}).$$

Thank you!

Construction of α:

Lemma

For every finite number of elements of $\mathcal{F}$, say $f, f_{n_1}, ..., f_{n_k}$, there exists some $s > l$ such that:

$$\mathcal{M} \models \exists y < t(\bar{v}, (a)_{i_0}) \forall i < s \left(\mathrm{P}(\bar{u}, m, \bar{v}, y, i, f) \wedge \bigwedge_{w=1}^{k} \mathrm{Q}(\bar{u}, m, \bar{v}, y, i, f_{n_w})\right).$$

Proof of Lemma:

Suppose not; i.e. there exists <u>the least</u> $k_0 \in \omega$ for which there exist some $f \in \mathcal{F}$ and k_0-many elements $f_{n_1}, ..., f_{n_{k_0}}$ of $\mathcal{F}$ such that for all $s > l$:

$$(1): \quad \mathcal{M} \models \forall y < t(\bar{v}, (a)_{i_0}) \forall i < s \neg \begin{pmatrix} \mathrm{P}(\bar{u}, m, \bar{v}, y, i, f) \wedge \\ \bigwedge_{w=1}^{k_0} \mathrm{Q}(\bar{u}, m, \bar{v}, y, i, f_{n_w}) \end{pmatrix}.$$

Construction of α:

To make things a little more clear, by taking a look at $\mathrm{P}(\bar{u}, m, \bar{v}, y, i, f)$ and $\mathrm{Q}(\bar{u}, m, \bar{v}, y, i, f_{n_w})$, statement (1) states that:

$$\textit{for all } y < t(\bar{v}, (a)_{i_0}) \textit{ and all } \epsilon \in M_0 \textit{ if } \overbrace{[f(\bar{u}, m, \epsilon)\downarrow] \rightarrow [f(\bar{v}, y, \epsilon)\downarrow]}^{\mathrm{P}(\bar{u}, m, \bar{v}, y, \epsilon, f)}\textit{, then}$$
$$\textit{there exists some } \xi \in M_0 \textit{ and some } w = 1, \dots, k_0 \textit{ s.t.}$$
$$\overbrace{[f_{n_w}(\bar{u}, m, \xi)\downarrow] \wedge [f_{n_w}(\bar{v}, y, \xi)\downarrow] \wedge f_{n_w}(\bar{u}, m, \xi) = f_{n_w}(\bar{v}, y, \xi)}^{\neg\mathrm{Q}(\bar{u}, m, \bar{v}, y, \xi, f_{n_w})}.$$

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By quantifying out $f_{n_w}(\bar{u}, m, \xi)$s from the above statement, it holds that:

$$\textit{There exists some } x \textit{ s.t. for all } y < t(\bar{v}, (a)_{i_0}) \textit{ and all } \epsilon \in M_0 \textit{ if } [f(\bar{u}, m, \epsilon)\downarrow] \to [f(\bar{v}, y, \epsilon)\downarrow]\textit{, then there exists some } \xi \in M_0 \textit{ and some } w = 1, ..., k_0 \textit{ s.t. } [f_{n_w}(\bar{u}, m, \xi)\downarrow] \wedge [f_{n_w}(\bar{v}, y, \xi)\downarrow] \wedge (x)_{<n_w,\xi>} = f_{n_w}(\bar{v}, y, \xi).$$

Construction of α:

Then we separate those subformulas of the above formula which contain parameters $\bar{u}$ and m. It turns out that the subsets defined in $\mathrm{H}^1(\mathcal{M}; I)$ with these subformulas can be coded by suitable elements of M_0. As a result, we will have a Σ_1-formula whose parameters are only $\bar{v}$ and some elements from M_0, say $(a)_{i_1}$ and $(a)_{i_2}$, which serve as codes the aforementioned subsets of $\mathrm{H}^1(\mathcal{M}; I)$; i.e. it holds that:

There exists some x s.t. for all $y < t(\bar{v}, (a)_{i_0})$ *and all* $\epsilon \mathrm{E} (a)_{i_1}$ *if* $[f(\bar{v}, y, \epsilon)\downarrow]$*, then there exists some* $\xi \mathrm{E} (a)_{i_2}$ *and some* $w = 1, ..., k_0$ *s.t.* $[f_{n_w}(\bar{v}, y, \xi)\downarrow] \wedge (x)_{<n_w, \xi>} = f_{n_w}(\bar{v}, y, \xi)$.

Construction of α:

Let:

$$g(\bar{v}) := \text{the smallest } x \text{ s.t. for all } y < t(\bar{v}, (a)_{i_0}) \text{ and all } \epsilon \mathrm{E}(a)_{i_1}, \text{ if } [f(\bar{v}, y, \epsilon)\downarrow], \text{ then there exists some } \xi \mathrm{E}(a)_{i_2} \text{ and some } w = 1, ..., k_0 \text{ s.t. } [f_{n_w}(\bar{v}, y, \xi)\downarrow] \wedge (x)_{<n_w, \xi>} = f_{n_w}(\bar{v}, y, \xi).$$

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Moreover, we define:

$$< o(\bar{v},y), h(\bar{v},y) > := \text{the smallest } < n_w, \xi > \text{ s.t } 1 \leq w \leq k_0 \text{ and } \xi \mathrm{E}(a)_{i_2} \text{ and } \epsilon \mathrm{E}(a)_{i_1}, \text{ if } [f(\bar{v},y,\epsilon)\downarrow], \text{ then } [f_{n_w}(\bar{v},y,\xi)\downarrow] \wedge (g(\bar{v}))_{<n_w,\xi>} = f_{n_w}(\bar{v},y,\xi).$$

Construction of α:

So it holds that:

$$\mathcal{M} \models \forall y < t(\bar{v}, (a)_{i_0})\, \forall \epsilon \mathrm{E}(a)_{i_1} \begin{pmatrix} [f(\bar{v}, y, \epsilon)\downarrow] \rightarrow \\ [< o(\bar{v}, y), h(\bar{v}, y) >\downarrow] \end{pmatrix}.$$

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So it holds that:

$$\mathcal{M} \models \forall y < t(\bar{v}, (a)_{i_0})\ \forall \epsilon \mathrm{E}(a)_{i_1} \left(\begin{array}{c} [f(\bar{v}, y, \epsilon) \downarrow] \rightarrow \\ [< o(\bar{v}, y), h(\bar{v}, y) > \downarrow] \end{array} \right).$$

Then by induction hypothesis:

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If $k_0 > 1$:

$$\mathcal{M} \models \forall y < t(\bar{v}, (a)_{i_0}) \forall i < s \neg \left(\begin{array}{c} \mathrm{P}(\bar{u}, m, \bar{v}, y, i, f') \wedge \\ \bigwedge_{w=2}^{k_0} \mathrm{Q}(\bar{u}, m, \bar{v}, y, i, f_{n_w}) \end{array} \right)$$; in which f' is:

$$f'(\diamondsuit, y) = \blacklozenge \Leftrightarrow f(\diamondsuit, y) = \blacklozenge \wedge [< o(\diamondsuit, y), h(\diamondsuit, y) >].$$

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But this contradicts the minimality of k_0.

Construction of α:

If $k_0 = 1$:

- $\mathcal{M}$ thinks that the cardinality of $A := \{h(\bar{u}, y) : \ \mathcal{M} \models (y < t(\bar{u}, (a)_{i_0}) \ \wedge \ [h(\bar{u}, y) \downarrow])\}$ is at most equal to the cardinality of $((a)_{i_1})_{\mathrm{E}}$.

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- By using the previous statements, we can build a coded 1-1 function $F \in M$ whose domain contains I and $F(I) \subset A$.

So again a contradiction is achieved by Σ_1-Pigeonhole Principle.

□