Tukey-top ultrafilters under UA

Tom Benhamou

Department of Mathematics at
Rutgers University

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Galvin's Theorem

In a paper by Baumgartner, Hajnal and Maté [2], the following theorem due to F. Galvin was published:

Theorem 1 (Galvin's Theorem)

Suppose that $\kappa^{<\kappa} = \kappa$. Then for every normal filter U over κ, and for any collection $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \in [U]^{\kappa^+}$ consisting of κ^+-many sets, there is a subcollection $\langle A_i \mid i \in I \rangle$, of size κ (i.e. $I \in [\kappa^+]^\kappa$) such that $\bigcap_{i \in I} A_i \in U$.

In particular, if *GCH* holds and κ is a regular cardinal then from κ^+-many clubs, one can always extract κ-many for which the intersection is a club.

Let us put this combinatorical/saturation property into a definition:

Definition 2 (Galvin's Property)

Let $\mathcal{F}$ be a filter over κ and $\mu \leq \lambda$. Denote by $Gal(\mathcal{F}, \mu, \lambda)$ the following statement:

$$\forall \langle A_i \mid i < \lambda \rangle \in [\mathcal{F}]^\lambda . \exists I \in [\lambda]^\mu . \bigcap_{i \in I} A_i \in \mathcal{F}$$

Example 3

1. Galvin's Theorem $\equiv$ If $\kappa^{<\kappa} = \kappa$ the $Gal(U, \kappa, \kappa^+)$ holds for every normal U over κ.

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2. If $\mu' \leq \mu \leq \lambda \leq \lambda'$ then $Gal(\mathcal{F}, \mu, \lambda) \Rightarrow Gal(\mathcal{F}, \mu', \lambda')$.
3. If (e.g.) $\mathcal{F}$ contains all the final segments and $\mu = cf(\kappa)$ then $\neg Gal(\mathcal{F}, \mu, \mu)$.

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$\Rightarrow$ Present a recent application of the Galvin property in the realm of the Tukey order (joint result with **N. Dobrinen**).

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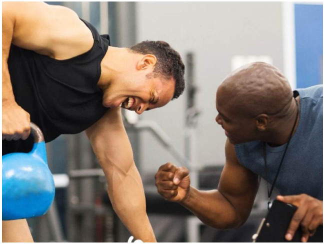
⇒ Present a recent application of the Galvin property in the realm of the Tukey order (joint result with **N. Dobrinen**).

⇒ Investigate the Galvin property under the ultrapower axiom (joint with **G. Goldberg**)

⇒ Continues the work [5, 7, 4, 8, 9] from the last couple of years of **Garti**, **Gitik**, **Poveda**, **Shelah**, and others.

Motivation



Slide from my CUNY talk in Fall 2021

Proposition 1

Suppose that $\kappa^{<\kappa} = \kappa$ *and let* F *be a normal filter over* κ. *The* F *satisfies The generalized Galvin property i.e. for every* $\langle X_i \mid i < \kappa^+\rangle \subseteq F$, *and each sequence* $\langle Z_i \mid i < \kappa^+\rangle$ *of subsets of* κ. *There is* $Y \subseteq \kappa^+$ *of cardinality* κ, *and* $\alpha \in \kappa^+ \setminus Y$ *such that*

1. $\bigcap_{i\in Y} X_i \in F$.
2. $[Z_\alpha]^{<\omega} \subseteq \bigcup_{i\in Y}[Z_i]^{<\omega}$.

Theorem 4

Suppose that $\mathcal{P}$ *is either Prikry or Magidor or Magidor-Radin or Radin or Prikry forcing with an ultrafilter satisfying the generalized Galvin Property. Let* $\underset{\sim}{Q}$ *be a quotient of* $\mathcal{P}$ *and* $G(\mathcal{P})$ *be a* V*-generic subset of* $\mathcal{P}$.
Then, the interpretation of $\underset{\sim}{Q}$ *in* $V[G(\mathcal{P})]$, *satisfies* κ^+-*c.c. there.*

The Tukey order

Definition 5

Let $(P, \leq_P), (Q, \leq_Q)$ be p.o's. Denote $P \leq_T Q$ if there is a *cofinal* function $f : Q \to P$, meaning that for every cofinal set $B \subseteq Q$ $(\forall q \in Q \, \exists b \in B \; q \leq_Q b)$ $f''B$ is cofinal in P. Also, $P =_T Q$ iff $P \leq_T Q \wedge Q \leq_T P$.

When U is a filter we are only considering $(U, \supseteq)$.

Example 6

- $\Rightarrow$ If $U \leq_{RK} W$ then $U \leq_T W$.
- $\Rightarrow$ $P \times Q = l.u.b_{\mathrm{T}}(P, Q)$.

Definition 7

We say that a partial order P is (μ, λ)-*Tukey-top* if the Tukey class of P is maximal among μ-directed-closed posets of cardinality at most λ.

Note that μ and λ can be inferred from P and so omitted from the definition.

Theorem 8 (Dobrinen-B.)

Let U be a μ-complete ultrafilter over κ. Then U is Tukey-top (wrt. $(\mu, 2^\kappa)$) if and only if $\neg Gal(U, \mu, 2^\kappa)$.

Ultrapower

Let U be a $\omega \leq \mu$-complete ultrafilter uniform over κ.

$\Rightarrow$ the ultrapower by U

$$j_U : V \to V^\kappa / U \simeq M_U, \;\; j_U(x) = [c_x]_U$$

Suppose that $W \in M_U$ is a σ-complete ultrafilter. Then

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Suppose that $W \in M_U$ is a σ-complete ultrafilter. Then

$\Rightarrow$ Internal to M_U, form $(M_W)^{M_U}$, and $j_W^{M_U} : M_U \to (M_W)^{M_U}$.

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$\Rightarrow$ The two step iterated ultrapower $j_W^{M_U} \circ j_U : V \to (M_W)^{M_U}$ is precisely the ultrapower by $j_{\sum_U W_\xi}$, where $W = [\xi \mapsto W_\xi]$, and

$$\sum_U W_\xi = \left\{X \subseteq \kappa \times \delta \mid \{\xi < \kappa \mid (X)_\xi \in W_\xi\} \in U\right\}$$

Summing the same ultrafilter is the (Fubini) product and the product of an ultrafilter with itself is called the (Fubini) power.

$$\begin{tikzcd}
 & M_{\sum_U W_\alpha} \\
V \arrow[ur, "j_{\sum_U W_\alpha}"] \arrow[dr, "j_U"'] & \\
 & M_U \arrow[uu, "j_W"']
\end{tikzcd}$$

$$W = [\alpha \mapsto W_\alpha]_U$$

The Rudin-Keisler, Rudin-Frólik orders

Definition 9

Suppose U, W are ultrafilters on λ, κ resp.

1. (Rudin-Keisler) Denote $U \leq_{RK} W$ if there is $f : \kappa \to \lambda$ such that $U = \{A \subseteq \lambda \mid f^{-1}{}'' A \in W\}$. We write $=_{RK}$ if f is injective.
2. (Rudin-Frólik) Denote by $U \leq_{RF} W$ if there is a sequence $\langle W_\xi \mid \xi < \lambda \rangle$ such that $W =_{RK} \sum_U W_\xi$.

$$
\begin{tikzcd}
 & M_W \\
V \arrow[ur, "j_W"] \arrow[dr, "j_U"'] & \\
 & M_U \arrow[uu, "j_{W^*}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M_W \\
V \arrow[ur, "j_W"] \arrow[dr, "j_U"'] & \\
 & M_U \arrow[uu, "k"']
\end{tikzcd}
$$

$$W^* = [\alpha \mapsto W_\alpha]_U$$

- $\Rightarrow$ $U \leq_{RF} W$ implies $U \leq_{RK} W$.
- $\Rightarrow$ Minimal ultrafilters in the Rudin-Keisler order are called *Ramsey* ultrafilters. (Equivalently, for every $f : [\kappa]^2 \to 2$ there is $X \in U$ such that $f \restriction [X]^2 = const.$)

Definition 10

An ultrafilter U is *irreducible* if it is RF-minimal among non-principal ultrafilters. Equivalently, there is no ultrapower embedding which factors j_U using an internal ultrapower.

Ramsey ultrafilters are irreducible.

Definition 11

An ultrafilter U over κ is called a *p*-point if every unbounded function *mod* U (namely, $f^{-1}[\mu] \notin U$ for all $\mu < \kappa$) is almost one-to-one *mod* U (namely if there is $X \in U$ such that for every $\gamma < \kappa$, $f^{-1}[\{\gamma\}] \cap X$ is bounded in κ).

Proposition 2

For κ-complete ultrafilters U over κ, TFRE:

1. *U is a p-point.*
2. *every κ-sequence in U had a pseudo intersection in U.*

How far can we push Galvin's Theorem?

$$[\kappa^{<\kappa} = \kappa \wedge U \text{ is normal}] \Rightarrow \ Gal(U, \kappa, \kappa^+)$$

The assumption $\kappa = \kappa^{<\kappa}$ was proven necessary by results of Abraham and Shelah [1] and later by Garti, Poveda, and B. [6].

Theorem 12 (Gitik-B.)

Suppose that $\kappa^{<\kappa} = \kappa$. Then

1. *([8] 2021) Every ultrafilter W which is Rudin-Keisler equivalent to a finite product of κ-complete p-point ultrafilters satisfies Gal(W, κ, κ^+).*
2. *([3] 2023) The same for an ultrafilter W which is Rudin-Keisler equivalent to an ultrafilter of the form:*

$$\sum_{U}(\sum_{U_{\alpha_1}} \cdots \sum_{U_{\alpha_1,\dots,\alpha_{n-1}}} (U_{\alpha_1,\dots,\alpha_n})\dots)$$

where U and each $U_{\alpha_1,\dots,\alpha_k}$ are p-point ultrafilters. Such an ultrafilter is called an n-fold sum of *p*-points.

We will see in a moment that the theorem above is optimal.

Theorem 13 (B. [3])

Suppose that $V = L[\mathbb{E}]$ where $L[\mathbb{E}]$ is an iterable Mitchell-steel model, and suppose that there is no measurable limit of superstrong cardinals, then every κ-complete ultrafilter is an n-fold sum of p-points and in particular has the Galvin property.

Theorem 14 (Goldberg-B. [10])

Assume UA plus every irreducible is Dodd sound, then for every κ-complete ultrafilter U over κ, U has the Galvin property iff U is an n-fold sum of p-points.

The assumptions of the theorem holds if $L[\mathbb{E}]$ is Mitchell-Steel iterable as proven by Schlutzenberg [15].

Theorem 15 (Gitik [11])

Starting from a measurable cardinal, it is consistent that there is an ultrafilter U with the Galvin property which is not an n-fold sum of p-points.

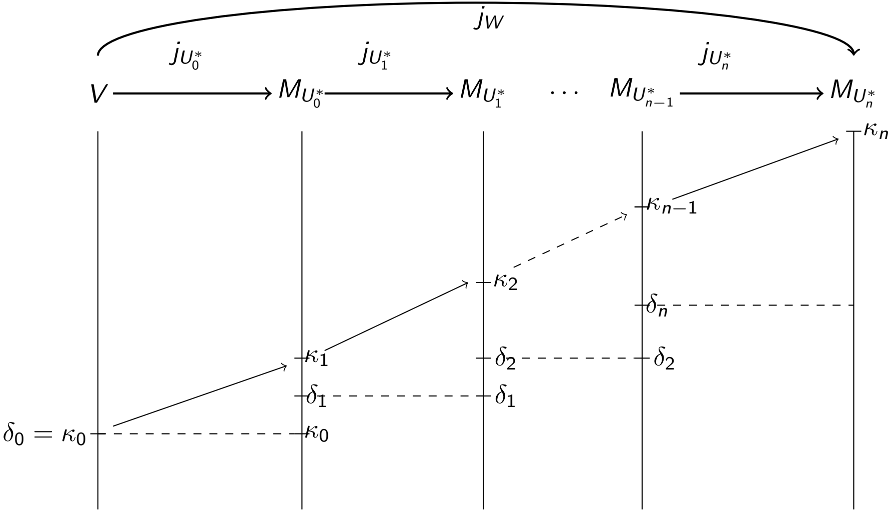
Theorem 14 (Goldberg-B.)

Assume UA plus every irreducible is Dodd sound, then for every κ-complete ultrafilter U over κ, U has the Galvin property iff U is an n-fold sum of p-points.

The proof

$$W = \sum_U (\sum_{U_{\alpha_1}} \cdots \sum_{U_{\alpha_1,\dots,\alpha_{n-1}}} U_{\alpha_1,\dots,\alpha_n})$$

$$U_0^* = U, \quad U_1^* = [\alpha_1 \mapsto U_{\alpha_1}]_{U_0^*}, \quad U_2^* = [\langle \alpha_1, \alpha_2 \rangle \mapsto U_{\alpha_1,\alpha_2}]_{\sum_U U_{\alpha_1}} \quad \dots$$

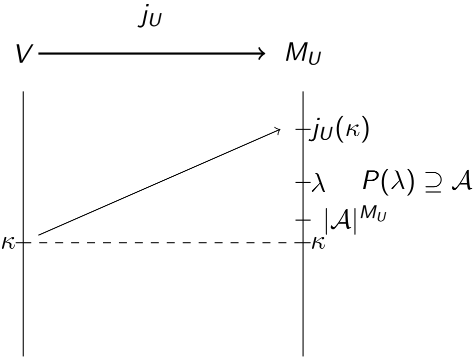


Diamond-like principles

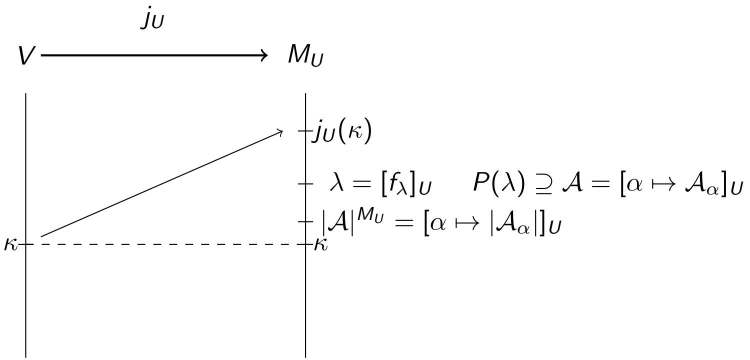
Definition 16

Let U be a σ-complete ultrafilter over κ. We say that $\diamondsuit^-_{\text{thin}}(U)$ holds iff there is $\mathcal{A} \in M_U$ and $\lambda < j_U(\kappa)$ such that:

1. $\{j_U(S) \cap \lambda \mid S \in P(\kappa)\} \subseteq \mathcal{A}$.
2. For every $f : \kappa \to \kappa$, $j_U(f)(|\mathcal{A}|^{M_U}) < \lambda$.



How is that a diamond-like principle?



The sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$ is a guessing sequence modulo U: for every $S \subseteq \kappa$, $\{\alpha < \kappa \mid S \cap f_\lambda(\alpha) \in \mathcal{A}_\alpha\} \in U$. The second condition implies that $|A_\alpha| < f_\lambda(\alpha)$ but it is much stronger. More accurately, in Puritz [13, 14] and Kanamori's [12] language of "skies and constellations", the cardinality of A should be in a lower "sky" then λ.

Definition 17

An ultrafilter U is λ-sound if the map $j^\lambda : P(\kappa) \to M_U$ defined by $j^\lambda(S) = j_U(S) \cap \lambda$ is in M_U. In particular $\{j_U(S) \cap \lambda \mid S \in P(\kappa)\} \in M_U$. U is called Dodd-sound if it is $[id]_U$-sound.

Normal ultrafilters are Dodd-sound which are in turn irreducible.

Proposition 3

If U is λ-sound for λ such that for all $f : \kappa \to \kappa$, $j_U(f)(\kappa) < \lambda$, then $\diamondsuit^-_{thin}(U)$ holds.

Proposition 4

For κ-complete ultrafilters U over κ, TFRE:

1. *U is a p-point.*
2. *there is a function f such that $j_U(g)(\kappa) \geq [id]_U$.*

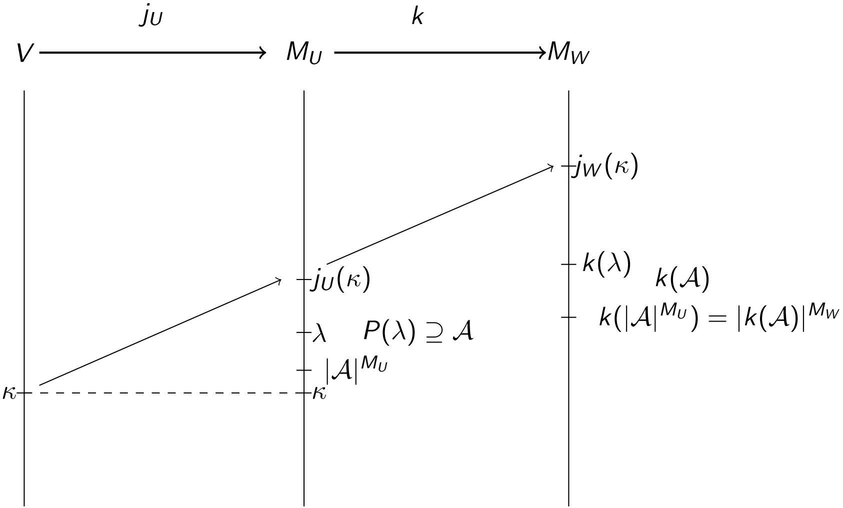
Corollary 18

If U is non p-point which is Dodd-sound then $\diamondsuit^-_{thin}(U)$ holds.

Proposition 5

Suppose $U \leq_{RK} W$. $\diamondsuit^{-}_{thin}(U) \Rightarrow \diamondsuit^{-}_{thin}(W)$.

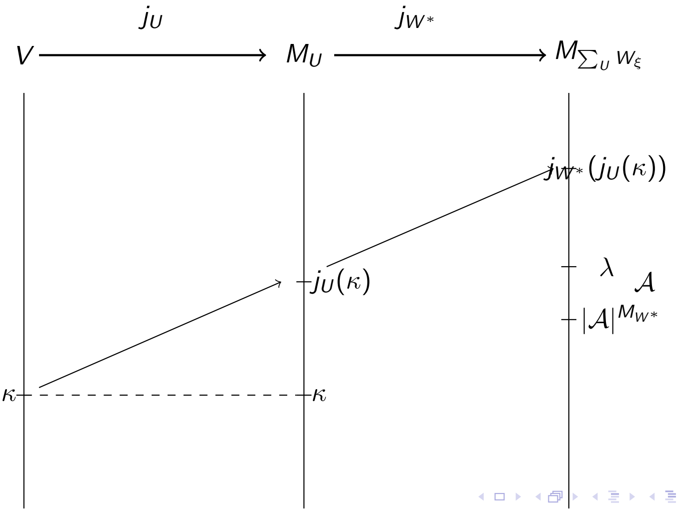
Proof.



Proposition 6

Suppose that U is an ultrafilter on $\lambda \leq \kappa$ and $\langle W_\xi \mid \xi < \lambda \rangle$ is a sequence of ultrafilters over κ such that for every ξ, $\diamondsuit^-_{thin}(W_\xi)$, then $\diamondsuit^-_{thin}(\sum_U W_\xi)$.

Proof.



The relevance of this principle to the Galvin property is the following:

Theorem 19

$\diamondsuit^{-}_{thin}(U)$ *implies* $\neg Gal(U, \kappa, 2^{\kappa})$.

The Ultrapower Axiom

The Ultrapower Axiom and its associated theory is due to G. Goldberg. It follows from weak comparison and therefore holds in every known canonical inner model.

Definition 20

The Ultrapower Axiom (UA) is the assertion that for every two σ-complete ultrafilters U, W, there are σ-complete ultrafilters $W^* \in M_U$ and $U^* \in M_W$ such that $(M_{U^*})^{M_W} = (M_{W^*})^{M_U}$ and $j_{U^*} \circ j_W = j_{W^*} \circ j_U$.

$$
\begin{tikzcd}
 & U^* \in M_W & \\
 & M_W \arrow[dr, dashed, "j_{U^*}"] & \\
V \arrow[ur, "j_W"] \arrow[dr, "j_U"'] & & (M_{W^*})^{M_U} = (M_{U^*})^{M_W} \\
 & M_U \arrow[ur, dashed, "j_{W^*}"'] & \\
 & W^* \in M_U &
\end{tikzcd}
$$

Theorem 21 (Goldberg (UA))

For every σ-complete ultrafilter U, every ascending sequence of ultrafilters $D_0 <_{RF} D_1 <_{RF} D_2 ... \leq_{RF} U$ is finite.

Theorem 22 (Goldbeg-B. (UA))

Assume that every irreducible is Dodd-sound. If W is a κ-complete ultrafilter over κ, then the following are equivalent:

1. *W has the Galvin property.*
2. $\neg\diamondsuit^{-}_{thin}(W)$.
3. *W is an n-fold sum of κ-complete p-points over κ*

Proof (Almost...)

- $\Rightarrow$ From previous results, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). We shall prove that (2) $\Rightarrow$ (3) and let W be an ultrafilter satisfying $\neg\diamondsuit^-_{\text{thin}}(W)$.
- $\Rightarrow$ Assume for example that $W = \sum_U W_\xi$ where U and all the W_ξ's are all irreducible.
- $\Rightarrow$ By our assumption, U and the W_ξ's are all Dodd-sound.
- $\Rightarrow$ If U is not a p-point, then it is a non-p-point Dodd sound ultrafilter and therefore $\diamondsuit^-_{\text{thin}}(U)$ which then implies that $\diamondsuit^-_{\text{thin}}(W)$, contradiction.
- $\Rightarrow$ If the set of W_ξ's which are non p-points is in U, then for each such ξ, $\diamondsuit^-_{\text{thin}}(W_\xi)$ and then $\diamondsuit^-_{\text{thin}}(\sum_U W_\xi)$), again a contradiction.

Thank you for your attention!

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Further Results

Theorem 23 (UA)

Assume that every irreducible ultrafilter is Dodd sound. For every σ-complete ultrafilter W over κ the following are equivalent:

1. *W has the Galvin property.*
2. $\neg\diamondsuit^{-}_{thin}(W)$.
3. *W is the D-sum of n-fold sums of κ-complete p-points over κ and D is a σ-complete ultrafilter on $\lambda < \kappa$.*

Theorem 24 (UA)

Assume that every irreducible ultrafilter is Dodd sound. Suppose κ is an uncountable cardinal that carries a κ-complete non-Galvin ultrafilter. Then the Ketonen least non-Galvin κ-complete ultrafilter on κ extends the closed unbounded filter.

Definition 25

κ is called *non-Galvin cardinal* if there are elementary embeddings $j\colon V \to M$, $i\colon V \to N$, $k\colon N \to M$ such that:

1. $k \circ i = j$.
2. $crit(j) = \kappa$, $crit(k) = i(\kappa)$.
3. ${}^{\kappa}N \subseteq N$ and ${}^{\kappa}M \subseteq M$
4. there is $A \in M$ such that $i''\kappa^+ \subseteq A$ and $M \models |A| < i(\kappa)$.

Theorem 26

Suppose that κ is a non-Galvin cardinal. Then there exists a κ-complete ultrafilter U over κ such that $\neg Gal(U, \kappa, \kappa^+)$.

Theorem 27 (UA)

Assume that every irreducible ultrafilter is Dodd sound. If there is a κ-complete non-Galvin ultrafilter on an uncountable cardinal κ, then there is a non-Galvin cardinal.