Subforcings of the Tree-Prikry Forcing

Tom Benhamou

Joint work with Moti Gitik and Yair Hayut

Department of Mathematics
Tel Aviv University

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Outline

1. Introduction
   - Background and Motivation
   - Basic definitions
2. Subforcings of the Tree-Prikry forcing
   - Known Results
   - Under very large cardinals
   - Under Minimal large cardinal assumption
   - Masterable forcing
   - Cardinality greater than κ
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Namely, in the **case of normal ultrafilter** U, the only projections and subforcings of the Prikry forcing are essentially the Prikry forcing with U. This situation changes drastically when considering the Prikry forcing suitable for non-normal ultrafilters: the Tree-Prikry forcing. We wish to examine the different possibilities under several large cardinal assumptions.

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For more information about projections, embeddings and boolean algebras see [12] or [1].

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If κ is κ-compact, every κ-distributive forcing of cardinality κ is a projection of a Tree-Prikry forcing.

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Theorem 8 (Gitik, Hayut, B.)

Let Q be the forcing shooting a club through the singulars below κ[a]. Assume that there is a κ-complete ultrafilter extending the filter $D(Q)$ of dense open subset of Q. Then either there is an inner model for $\exists\lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) \geq \kappa^+$.

[a] Thus Making κ not Mahlo. It is $< \kappa$-strategically closed.

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From minimal large cardinal assumption

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Assume GCH and let κ be a measurable cardinal. There is a cofinality preserving forcing extension $V \subseteq N$ and an ultrefilter $W \in N$ such that forcing with $P_T(W)$[a] over N adds a κ-Cohen real.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = W$.

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$$f = fc_{n_0} \cup \cup_{n_0 < n < \omega} f_{c_n} \restriction [c_{n-1}, c_n) \in N[G]$$

is N-generic for $Add(\kappa, 1)$. □

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As we have seen, moving to the class of $< \kappa$-strategically closed forcing already require larger cardinals. Nonetheless, D. Velleman proved [13] that if there is a Jensen square to κ, then every $< \kappa-$strategically closed forcing is actually $\kappa-$strategically closed forcing. This provides a "maximal" forcing notion to the class of $< \kappa$-strategically closed forcing: The forcing which adds a Jensen square.

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Question

Under the minimal assumption that κ is measurable. What is the class of forcing $\mathcal{P}$ which can be intermediate to a Tree-Prikry extension?

Masterable forcing

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For example, $Add(\kappa, 1)$ is masterable by taking $\underset{\sim}{\mathbb{R}}$ as the trivial forcing.

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- Prepare the universe V by forcing the Easton support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ where $\underset{\sim}{\mathbb{Q}}_\alpha$ is either the trivial forcing, if α is not inaccessible, or the lottery sum of all $<\alpha$-strategically closed forcing notions of size α.

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- Given $\mathbb{Q} \in \mathcal{N}_\kappa^{V[G]}$, by elementarity $j_1^*(\mathbb{Q}) \in \mathcal{N}_{\kappa_1}^{M[G_1]}$, is also masterable where $\kappa_1 = j_1(\kappa)$.

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- By condition (2) of masterable, there is a further forcing $\underset{\sim}{\mathbb{R}}$ with properties $(2a), (2b)$.

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- Given $\mathbb{Q} \in \mathcal{N}_\kappa^{V[G]}$, by elementarity $j_1^*(\mathbb{Q}) \in \mathcal{N}_{\kappa_1}^{M[G_1]}$, is also masterable where $\kappa_1 = j_1(\kappa)$.
- By condition (2) of masterable, there is a further forcing $\underset{\sim}{\mathbb{R}}$ with properties $(2a), (2b)$.
- By $(2b)$, the forcing is $< \kappa_1-$str.cl., hence by GCH, in $V[G]$ we can construct an $M[G_1]$-generic filter $H := G_{j_1^*(\mathbb{Q})} * G_{\underset{\sim}{\mathbb{R}}}$.

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- By (2a), in $M[G_1][H]$ there is a generic elementary embedding $k : M[G_1][G_{j_1^*(\mathbb{Q})}] \to N_0[k(G_{j_1^*(\mathbb{Q})})]$ and $G_{j_1^*(\mathbb{Q})} = k'' G_{j_1^*(\mathbb{Q})} \subseteq k(G_{j_1^*(\mathbb{Q})})$.

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Outline

1. Introduction
   - Background and Motivation
   - Basic definitions
2. Subforcings of the Tree-Prikry forcing
   - Known Results
   - Under very large cardinals
   - Under Minimal large cardinal assumption
   - Masterable forcing
   - Cardinality greater than κ
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Starting from a measurable cardinal, it is concictent that there is a non-Galvin ultrafilter U such that forcing $\mathbb{P}_T(U)$ adds a generic for $Add(\kappa, \kappa^+)$.

References I

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Thank you for your attention!