Intermediate Prikry-type models, quotients, and the Galvin property

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Theorem 2 (Gitik, Kanovei, Koepke, 2010 [10])

Let U be a normal measure over κ and $G \subseteq \mathbb{P}(U)$ be a V-generic filter producing the Prikry sequence $C_G := \{\kappa_n \mid n < \omega\}$. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

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Every such model is of the form $M = V[A]$ for some set $A \in V[G]$. By theorem 2, $M = V[C]$ for some subsequence C of the Prikry sequence. By the Mathias criteria[15], C is itself a Prikry sequence. □

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What forcings $\mathbb{P}$, have (consistently) generic extension intermediate to a generic extension by Magidor-Radin forcing or the Tree-Prikry forcing?.

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1. Background
2. Magidor-Radin Forcing
   - The Forcing Notion
   - Examples & Main result
   - The Proof
     - Short Sequence
     - Subsets of κ(Proof omitted)
     - The remaining cases
3. The quotient forcing and Galvin's property
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   - κ^+-c.c. of quotients and the Galvin property
4. The Tree-Prikry forcing
5. References

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Our forcing notations are in Israeli style i.e. $p \leq q$ means that q is stronger.

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2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \cap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.

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   The order is define as follows,
   $p := \langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle \leq q := \langle\langle\beta_1, B_1\rangle, ..., \langle\beta_m, B_m\rangle, \langle\kappa, B\rangle\rangle$ iff:

   $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:
   1. If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\beta_{i_r} = \alpha_r$ and $B_{i_r} \subseteq A_r$.
   2. Otherwise let $1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then:
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   If $p \leq q$ and in addition $n = m$, denote it by $p \leq^* q$.

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8. For every V-regular cardinal α, if $cf^{V[G]}(\alpha) < \alpha$ then $\alpha \in Lim(C_G)$.
9. If $\alpha \in C_G \cup \{\kappa\}$, $0 < o^{\vec{U}}(\alpha)$ and $cf(o^{\vec{U}}(\alpha)) < \alpha^+$ then $cf^{V[G]}(\alpha) < \alpha$.

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1. $C_G = \{\nu \mid \exists\ A \exists p \in G\ s.t.\ \langle \nu, A\rangle \in p\}$ is a club. $\mathrm{otp}(C_G) = \min\{\kappa, \omega^{o^{\vec{U}}(\kappa)}\}$.
2. If $\alpha \in C_G$, $o^{\vec{U}}(\alpha) = 0$ iff α is successor in C_G.
3. Satisfies k^+-c.c. (even (κ, κ)-centered) and the Prikry condition.
4. If $\delta \in Lim(C_G) \cup \{\kappa\}$ and $A \in \cap_{\xi < o^{\vec{U}}(\delta)} U(\delta, \xi)$, $\exists \delta' < \delta$ such $(\delta', \delta) \cap C_G \subseteq A$.
5. For $\vec{U}$-measurable λ, $\mathbb{M}[\vec{U}]$ naturally factors to the a Magidor forcing up to λ, denoted by $\mathbb{M}[\vec{U}] \restriction \lambda$ and above λ, denoted by $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$. The first part is of cardinality 2^λ and the second has $\leq^*$-closure degree much above 2^λ.
6. If $A \subseteq V_\alpha$, then $A \in V[C_G \cap \lambda]$, where $\lambda = \max(Lim(C_G) \cap \alpha + 1)$.
7. $\mathbb{M}[\vec{U}]$ preserves cardinals.
8. For every V-regular cardinal α, if $cf^{V[G]}(\alpha) < \alpha$ then $\alpha \in Lim(C_G)$.
9. If $\alpha \in C_G \cup \{\kappa\}$, $0 < o^{\vec{U}}(\alpha)$ and $cf(o^{\vec{U}}(\alpha)) < \alpha^+$ then $cf^{V[G]}(\alpha) < \alpha$.
10. If $\alpha \in C_G \cup \{\kappa\}$ and $cf(o^{\vec{U}}(\alpha)) \geq \alpha^+$ then α is regular in $V[G]$.

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Assume that $o^{\vec{U}}(\kappa) = \omega$, thus $\mathrm{otp}(C_G) = \omega^\omega$. Consider the intermediate extension $V[\{C_G(\omega^n) \mid n < \omega\}]$ it is a diagonal Prikry generic extension for the sequence of measures $\langle U(\kappa, n) \mid n < \omega\rangle$.

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Suppose $o^{\vec{U}}(\kappa) = \kappa$, then $C_G = \{C_G(\alpha) \mid \alpha < \kappa\}$. In $V[G]$, define $\alpha_0 = C_G(0)$, and $\alpha_{n+1} = C_G(\alpha_n)$. Since $\{\alpha < \kappa \mid o^{\vec{U}}(\alpha) < \alpha\}$ is measure-one, $\{\alpha_n \mid n < \omega\}$ is a cofinal ω-sequence in κ. Also, it satisfy the Mathias criteria [3] for the Tree-Prikry forcing of the measures $\langle U(\kappa, \alpha) \mid \alpha < \kappa\rangle$.

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Clearly all these example are Prikry-Type extensions.

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(*IH*) *For every $\delta < \kappa$, any coherent sequence $\vec{W}$ with maximal measurable δ and any set $A \in V[H]$ for $H \subseteq \mathbb{M}[\vec{W}]$, there is $C \subseteq C_H$, such that $V[A] = V[C]$.*

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Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and any set $A \in V[G]$, there is $C \subseteq C_G$ such that $V[A] = V[C]$.

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As a corollary of this, we obtain the first step toward a classification:

Corollary 11

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Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and any set $A \in V[G]$, there is $C \subseteq C_G$ such that $V[A] = V[C]$.

As a corollary of this, we obtain the first step toward a classification:

Corollary 11

Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic filter producing the Magidor sequence C_G. Assume that $\forall \alpha \in C_G \cup \{\kappa\}. o^{\vec{U}}(\alpha) < \alpha^+$. Then for every $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$.

The Main Result

As we have seen from the examples, it is not clear which are the forcings such that the models $V[C]$ are generic extensions of. In [4], we restrict the order of κ to be below κ and define a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_f[\vec{U}]$. This class is basically a Magidor forcing adding elements from measures prescribed by the function f. We then prove that the intermediate model must be finite iterations of such forcings.

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If time permits we will discuss it later. Let us sketch some of the ideas from the proof of 10.

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Proposition 1

It suffices to prove that for sets of ordinals X, $V[X] = V[C]$ for some $C \subseteq C_G$.

Proof

If A is any set, then by [11, Thm. 15.42] there is a forcing $\mathbb{Q} \in V$ and a generic $H \subseteq \mathbb{Q}$ such that $V[A] = V[H]$. Let $\lambda = |\mathbb{Q}|$, $f : \mathbb{Q} \leftrightarrow \lambda \in V$ a bijection and $f''H = X \subseteq \lambda$. Then $V[H] = V[X]$, and by assumption there is $C \subseteq C_G$ such that $V[X] = V[C]$, implying $V[A] = V[X] = V[C]$. □

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Lemma 12

If $A \subseteq V, A \in V[G]$, $|A| < \kappa$, then there is $C \subseteq C_G$ such that $V[A] = V[C]$.

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A tree $T \subseteq [\kappa]^{<\omega}$ is called a $\vec{U}$-fat tree, if $ht(T) < \omega$ and for every $t \in T$, either or $succ_T(t) := \{\alpha < \kappa \mid t^\frown\alpha \in T\} \in U(\beta, i)$ for some $\beta \leq \kappa$ and $i < o^{\vec{U}}(\beta)$, or t is a maximal element of the tree. Denote the set of Maximal elements by $mb(T)$.

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Suppose that $p \in \mathbb{M}[\vec{U}]$ and $D \subseteq \mathbb{M}[\vec{U}]$ is a dense open subset. Then there is $p \leq^* p^*$ and a $\vec{U}$-fat tree T, such that for every $\vec{b} \in mb(T)$, $p^{*\frown}\vec{b} \in D$.

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Lemma 14 ([5])

Let T be a $\vec{U}$-fat tree and $f : mb(T) \to B$ where B is any set. Then there is a $\vec{U}$-fat tree $T' \subseteq T$, with $ht(T') = ht(T)$ and $I \subseteq \{1, ..., ht(T)\}$ such that for any $t, t' \in mb(T')$: $t \restriction I = t' \restriction I \Leftrightarrow f(t) = f(t')$.

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Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega\rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A.

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Assume for example that $A = \{a_n \mid n < \omega\}$ and let $\langle \underset{\sim}{a}_n \mid n < \omega \rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for A. Let $p \in \mathbb{M}[\vec{U}]$, for each n apply the Strong Prikry property to find $p \leq^* p_n$ and a $\vec{U}$-fat tree T_n such that for every $\vec{\beta} \in mb(T_n)$, there is γ $p_n^\frown \vec{\beta} \Vdash \underset{\sim}{a}_n = \gamma$. Denote by $f_n(\vec{\beta}) = \gamma$. Apply the previous lemma, shrink the tree T_n to T_n^* and find $I_n \subseteq \{1, ..., ht(T_n)\}$. By $\leq^*$-closure, find a single p_ω such that $p_n \leq^* p_\omega$. If necessary, extend $p_\omega \leq^* p^*$ so that the set $\{p^{*\frown} t \mid t \in mb(T_n^*)\}$ is pre-dense above p^* for every $n < \omega$. By density find such $p^* \in G$. Then there is a branch $D_n \in mb(T_n^*)$ such that $p^{*\frown} D_n \in G$. Since $(\underset{\sim}{a}_n)_G = a_n$ it follows that $f_n(D_n) = a_n$, define $C = \cup_{n<\omega}(D_n) \restriction I_n$. Let us prove that $V[A] = V[C]$: In $V[C]$ we can construct the sequence $\langle (D_n) \restriction I_n \mid n < \omega \rangle$, then use AC to find branchs $\langle D'_n \mid n < \omega \rangle$ such that $D'_n \in mb(T_n^*)$ and $(D'_n) \restriction I_n = (D_n) \restriction I_n$ hence $f_n(D'_n) = f_n(D_n) = a_n$ and $A = \{f_n(D'_n) \mid n < \omega\} \in V[C]$.
In $V[A]$ we can calculate each $(D_n) \restriction I_n$, by taking any $D'_n \in f_n^{-1}(a_n)$. Since $f_n(D_n) = a_n = f_n(D'_n)$, it follows that $(D_n) \restriction I_n = (D'_n) \restriction I_n$. We conclude that $C = \cup_{n<\omega}(D'_n) \restriction I_n \in V[A]$ and $V[A] = V[C]$. $\square$

A Property of the Magidor Generic Club

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As a consequence, we obtain Fuchs result to this variation of Magidor-Radin forcing:

Corollary 16

Let G, G' be V-generic filters for $\mathbb{M}[\vec{U}]$. If $G' \in V[G]$ then $C_{G'} \setminus C_G$ is finite. In particular $V[G] = V[G']$ iff $C_G \Delta C_{G'}$ is finite.

Outline

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Proof.

Since κ is singular in $V[G]$ then $cf^{V[G]}(\lambda) < \kappa$ and by $\kappa^+ - c.c.$ of $\mathbb{M}[\vec{U}]$, $\nu := cf^V(\lambda) \leq \kappa$. Fix $\langle \gamma_i | \ i < \nu \rangle \in V$ cofinal in λ.

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Assume $o^{\vec{U}}(\kappa) < \kappa^+$, *and* $cf^{V[G]}(\lambda) \leq \kappa$, *then* $\exists C \subseteq C_G$ *s.t.* $V[A] = V[C]$.

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Since κ is singular in $V[G]$ then $cf^{V[G]}(\lambda) < \kappa$ and by $\kappa^+ - c.c.$ of $\mathbb{M}[\vec{U}]$, $\nu := cf^V(\lambda) \leq \kappa$. Fix $\langle \gamma_i | \ i < \nu \rangle \in V$ cofinal in λ. Work in $V[A]$, for every $i < \nu$ find $d_i \subseteq \kappa$ such that $V[d_i] = V[A \cap \gamma_i]$. By induction, there exists $C^* \subseteq C_G$ such that $V[\langle d_i \mid i < \nu \rangle] = V[C^*]$, therefore $C^* \in V[A]$ and $\forall i < \nu \ A \cap \gamma_i \in V[C^*]$. This does not mean that $A \in V[C^*]$.

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Assume that $\theta := cf^{V[G]}(\lambda) > \kappa$. To find the desired $C \subseteq C_G$, it is tempting take a cofinal sequence α_i in $V[A]$, apply the induction hypothesis to $A \cap \alpha_i$ for every $i < \theta$ to obtain $C_i \subseteq C_G$ such that $V[C_i] = V[A \cap \alpha_i]$ and take $C = \cup_{i<\theta} C_i$. However there are three problems here:

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Again let $o^{\vec{U}}(\kappa) = \delta_0$, $o^{\vec{U}}(\delta_0) = 1$, and a generic G such that $\mathrm{otp}(C_G) = C_G(\omega) = \delta$. Let

$$D = \{C_G(C_G(n)) \mid n < \omega\} \text{ and } E = \{C_G(\alpha) \mid \omega \leq \alpha < C_G(\omega)\} \setminus D$$

Then $D \cup E = \{C_G(\alpha) \mid \omega \leq \alpha < C_G(\omega)\}$, hence in $V[D \cup E]$, $C_G(\omega)$ is still measurable. On the other hand, from D, we can reconstruct $\langle C_G(n) \mid n < \omega \rangle$ as $o^{\vec{U}}(C_G(C_G(n))) = C_G(n)$. So it if impossible that $D \in V[D \cup E]$.

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To deal with problem ❶, we need to somehow make the choice of the C_i's inside the model $V[A]$. This seems impossible as it involves referring to C_G which is not available in $V[A]$.

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To deal with problem ①, we need to somehow make the choice of the C_i's inside the model $V[A]$. This seems impossible as it involves referring to C_G which is not available in $V[A]$. However, consider the following definition:

Mathias Sets

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Let $X_A := \{\nu < \kappa \mid cf^{V[A]}(\nu) < cf^{V}(\nu) = \nu\}$. A set $D \in V[A]$ is called a *Mathias set*, if :

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The following proposition enables us to refer to subsets of C_G in $V[A]$:

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The following proposition enables us to refer to subsets of C_G in $V[A]$:

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Proof.

The Direction $D \subseteq^* C_G$ implies that D is a Mathias set, is a standard density argument of C_G. For the other direction, we can use lemma 15. □

Overcoming the First Problem

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Let us use Mathias sets in order to overcome the first obstacle: We use induction hypothesis and the axiom of choice to find Mathias sets D_i such that $V[D_i] = V[A \cap \alpha_i]$ and additionally $\langle D_{\alpha_i} \mid i < \theta \rangle \in V[A]$.

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Theorem 19

Let $\aleph_0 < \kappa$ be a strong limit cardinal, and $\mu > \kappa$ be regular. Let $\langle D_\alpha \mid \alpha < \mu \rangle$ be any $\subseteq^*$-increasing sequence of subsets of κ. Then the sequence $=^*$-stabilizes i.e. there is $\alpha^* < \mu$ such that for every $\alpha^* \leq \alpha < \mu$, $D_\alpha =^* D_{\alpha^*}$.

Proof of theorem 19

Let α^* be a stabilization point, then $V[D_{\alpha^*}]$ includes all the initial of A.

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$$(*) \qquad \text{For every } \alpha^* \leq \beta_1 < \beta_2 < \mu. \quad |D_{\beta_1} \Delta D_{\beta_2}| \leq \omega$$

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Also $cf(\kappa) = \omega$, since for any distinct $\beta_1, \beta_2 \in Y \setminus \alpha^*$, $|D_{\beta_1} \Delta D_{\beta_2}| = \aleph_0$, and cannot be bounded.

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Also $cf(\kappa) = \omega$, since for any distinct $\beta_1, \beta_2 \in Y \setminus \alpha^*$, $|D_{\beta_1} \Delta D_{\beta_2}| = \aleph_0$, and cannot be bounded. Let $\langle \eta_n \mid n < \omega \rangle$ be cofinal in κ. Define a partition $f : [Y \setminus \alpha^*]^2 \to \omega$: For any $i < j$ in $Y \setminus \alpha^*$, let $f(i,j) = n_{i,j} < \omega$ such that $(D_{\alpha_i} \setminus \eta_{n_{i,j}}) \subseteq (D_{\alpha_j} \setminus \eta_{n_{i,j}})$. It is well defined as $D_{\alpha_i} \setminus D_{\alpha_j}$ is finite.

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Finally, to resolve problem 3 . We will show that there are no fresh subsets with respect to the models $V[C] \subseteq V[G]$ i.e. if $\forall \alpha < \sup(A)$, $A \cap \alpha \in V[C]$ then $A \in V[C]$. The forcing completing $V[C]$ to $V[G]$ is the quotient and from the following theorems we can deduce that this quotient does not add fresh subsets.

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Theorem 20

Every quotient of $\mathbb{M}[\vec{U}]$ *is* κ^+*-c.c. in* $V[G]$.

Overcoming the Third Problem

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Theorem 21 (No Fresh Subsets of cofinality λ)

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Let $W \models ZFC$ and $\mathbb{P} \in W$ a forcing notion. Let $T \subseteq \mathbb{P}$ be any W-generic filter and θ is a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is θ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets with respect to W of cardinals λ such that $\theta = cf(\lambda)$.

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Assume otherwise and let $A \in W[T]$ be a fresh subset of λ. Pick a name $\underset{\sim}{A}$ for A and work within $W[T]$.

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Outline

The quotient forcing I

To finish the proof it remains to show that quotients are κ^+-c.c. Before, let us recall some basic facts about the quotient.

Definition 22

Let $\mathbb{P}, Q$ be forcing notions. A function $\tau : \mathbb{P} \to \mathbb{Q}$ is a projection iff τ is order preserving, $Im(\tau)$ is dense, and

$$\forall p \in \mathbb{P}. \forall q \geq \tau(p). \exists p' \geq p. \pi(p') \geq q$$

Definition 23

Let $\mathbb{P}, \mathbb{Q} \in V$ be forcing notions, $\tau : \mathbb{P} \to \mathbb{Q}$ be any projection and let $H \subseteq \mathbb{Q}$ be V-generic. Define *the quotient forcing* $\mathbb{P}/H = \tau^{-1''} H$. Also if $G \subseteq \mathbb{P}$ is a V-generic filter, *the projection of* G is the filter

$$\tau_*(G) := \{q \in \mathbb{Q} \mid \exists p \in G. q \leq_{\mathbb{Q}} \tau(p)\}$$

The quotient forcing II

Proposition 6

Let $\tau : \mathbb{P} \to \mathbb{Q}$ *be a projection, then:*

1. *If* $G \subseteq \mathbb{P}$ *is* V*-generic then* $\tau_*(G)$ *is* V*-generic filter for* $\mathbb{Q}$
2. *If* $G \subseteq \mathbb{P}$ *is* V*-generic then* $G \subseteq \mathbb{P}/\tau_*(G)$ *is* $V[\tau_*(G)]$*-generic filter.*
3. *If* $H \subseteq \mathbb{Q}$ *is* V*-generic and* $G \subseteq \mathbb{P}/H$ *is* $V[H]$*-generic, then* $\tau_*(G) = H$ *and* $G \subseteq \mathbb{P}$ *is* V*-generic.*

Definition 24

Let $\mathbb{P}$ be a forcing notion and $\underset{\sim}{D}$ be a $\mathbb{P}$-name for a subset of κ. Define $\mathbb{P}_{\underset{\sim}{D}}$, the complete subalgebra of regular open cuts $\langle RO(\mathbb{P}), \leq_B \rangle$ [a] generated by the set $X = \{||\alpha \in \underset{\sim}{D}|| \mid \alpha < \kappa\}$.

[a] The order $\leq_B$ is in the standard position of Boolean algebras orders i.e. $p \leq_B q$ means $p \Vdash q \in \hat{G}$.

The quotient forcing III

Definition 25

Define the function $\pi : \mathbb{P} \to \mathbb{P}_{\underset{\sim}{D}}$ by $\pi(p) = \inf\{b \in \mathbb{P}_{\underset{\sim}{D}} \mid p \leq_B b\}$.

It not hard to check that π is a projection. Let G be V-generic for $\mathbb{P}$ and $D \subseteq \kappa$ the interpretation of $\underset{\sim}{D}$ under G i.e. $\underset{\sim}{D}_G = D$. Denote by $H = \pi_*(G)$ the V-generic filter for $\mathbb{P}_{\underset{\sim}{D}}$ induce, then $\widetilde{V}[D] = V[H]$ (see for example [11, 15.42]). In fact

$$D = \{\alpha < \kappa \mid ||\alpha \in \underset{\sim}{D}|| \in X \cap H\}$$

As for the other direction, any generic filter H is definable and uniquely determined (see [11, Lemma 15.40]) by the set

$$X \cap H = \{||\alpha \in \underset{\sim}{D}|| \mid \alpha \in D\}$$

We sometimes abuse notation by defining $\mathbb{P}/D = \mathbb{P}/\pi_*(G)$. It is important to note that $\mathbb{P}/D$ depends on the choice of the name $\underset{\sim}{D}$.

The quotient forcing IV

Proposition 7

For every $q \in \mathbb{P}$, and let G be V-generic. Denote by $H = \pi_*(G)$. Then $q \in \mathbb{P}/H$ iff there is a V-generic $G' \subseteq \mathbb{P}$ such that $q \in G'$ and $\pi_*(G') = H$.

Note that since $\pi_*(G')$ is uniquely determined by $X \cap \pi_*(G')$, the requirement that $\pi_*(G') = \pi_*(G)$ is equivalent to $\underset{\sim}{D}_{G'} = \underset{\sim}{D}_G$.

Proof.

Let $q \in \mathbb{P}/H$, $G' \subseteq \mathbb{P}/H$ be any $V[H]$-generic with $q \in G'$. Then by proposition 6.3, $G' \subseteq \mathbb{P}$ is a V-generic filter and $\pi_*(G') = \pi_*(G) = H$. For the other direction, if $q \in G'$ for some $G' \subseteq \mathbb{P}$ such that $\pi_*(G') = H$, then $\pi_*(G') = \pi_*(G)$. Since, $\pi(q) \in \pi(G') = \pi_*(G)$, then $a \in \pi^{-1''} H =: \mathbb{P}/H$. □

Let us turn to the proof of κ^+-c.c.:

The quotient forcing V

Theorem 26

Let $\pi : \mathbb{M}[\vec{U}] \to \mathbb{P}$ be a projection and $G \subseteq \mathbb{M}[\vec{U}]$ be V-generic and $H = \pi_*(G)$ be the induced generic for $\mathbb{P}$. Then $V[G] \models \mathbb{M}[\vec{U}]/H$ is κ^+-c.c.

Note that the standard argument for κ^+-c.c. does not work: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an antichain in $\mathbb{M}[\vec{U}]/H$. Each p_i is of the form $p_{i,\downarrow}^\frown \langle \kappa, A_i \rangle$. Since κ^+ is still regular in $V[G]$, there are $i \neq j$ such that $p_{i,\downarrow} = p_{j,\downarrow}$. Hence $p_{i,\downarrow}^\frown \langle \kappa, A_i \cap A_j \rangle \geq p_i, p_j$. However, $p_{i,\downarrow}^\frown \langle \kappa, A_i \cap A_j \rangle$ might not be in $\mathbb{M}[\vec{U}]/H$:

Example 27

In Prikry forcing, let $C = \{C_G(2n) \mid n < \omega\}$. Conditions in $P(U)/H$ are $\langle \alpha_0, ..., \alpha_n, A \rangle$ such that:

1. $\alpha_{2i} = C_G(2i)$.
2. For $m > n/2$, $C_G(2m) \in A$.
3. For $m > n/2$, $(C_G(2m-2), C_G(2m)) \cap A \neq \emptyset$.

The quotient forcing VI

The third condition might fail when intersecting large sets.

Proof of 26: Assume otherwise, and let $\langle p_i \mid i < \kappa^+ \rangle \in V[G]$ be an anthichain in $\mathbb{M}[\vec{U}]/H$. Let $\langle \underset{\sim}{p}_i \mid i < \kappa^+ \rangle$ be a sequence of $\mathbb{M}[\vec{U}]$-names for them and $r \in G$ such that

$$r \Vdash \langle \underset{\sim}{p}_i \mid i < \kappa^+ \rangle \text{ is an antichain in } \mathbb{M}[\vec{U}]/\underset{\sim}{H}$$

Work in V, for every $i < \kappa^+$, let $r \leq r_i \in \mathbb{M}[\vec{U}]$ and $\xi_i \in \mathbb{M}[\vec{U}]$ be such that $r_i \Vdash \underset{\sim}{p}_i = \xi_i$.

Lemma 28

There is $q_i \geq \xi_i$ such that $\forall q \geq q_i \exists r'' \geq r_i \ r'' \Vdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$

Proof of Lemma: Otherwise, for every $q \geq \xi_i$, there is $q' \geq q$ such that every $r'' \geq r_i$, $r'' \nVdash q' \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$. In particular, the set

$$E = \{q \geq \xi_i \mid \forall r'' \geq r_i . r'' \nVdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}\}$$

is dense above ξ_i. To obtain a contradiction, let G' be any generic for $\mathbb{M}[\vec{U}]$ such that $r_i \in G'$ and denote $H' = (\underset{\sim}{H})_{G'} = \pi_*(G')$. Since $r_i \geq r$, $r \in G'$ and therefore $\xi_i = (\underset{\sim}{p}_i)_{G'} \in \mathbb{M}[\vec{U}]/H'$. Then there is a V-generic filter G'' for $\mathbb{M}[\vec{U}]$ such that $\xi_i \in G''$ and $\pi_*(G'') = H'$. By density of E, there is $\xi_i \leq q \in E \cap G''$ and in particular, $q \in \mathbb{M}[\vec{U}]/H'$. Thus, there is $r_i \leq r'' \in G'$ such that $r'' \Vdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$, contradicting $q \in E$.$\square_{Lemma}$

For every $i < \kappa^+$ fix $q_i \geq \xi_i$ such that

$$(*)_i \qquad \forall q \geq q_i . \exists r'' \geq r_i . r'' \Vdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$$

Denote by $q_i = \langle t_{i,1}, ..., t_{i,n_i}, \langle \kappa, A(q_i) \rangle \rangle$ and $r_i = \langle s_{i,1}, ..., s_{i,m_i}, \langle \kappa, A(r_i) \rangle \rangle$. Find $X \subseteq \kappa^+$ such that $|X| = \kappa^+$ and $\vec{t} = \langle t_1, .., t_n \rangle, \vec{s} = \langle s_1, ..., s_m \rangle$ such that for every $i \in X$, $\langle t_{i1}, ..., t_{in_i} \rangle = \langle t_1, .., t_n \rangle$, and $\langle s_{i1}, ..., s_{im_i} \rangle = \langle s_1, ..., s_m \rangle$. This means that for every $i \in X$, $q_i = \vec{t}^\frown \langle \kappa, A(q_i) \rangle$ and $r_i = \vec{s}^\frown \langle \kappa, A(r_i) \rangle$. Let $q = \vec{t}^\frown \langle \kappa, A(q_i) \cap A(q_j) \rangle$, then by $(*)_i$ there is $r' \geq r_i$ such that r' forces $q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$. This means that r' must be incompatible with r_j. Otherwise, there would be $r'' \geq r', r_i, r_j$, which forces contradictory information. Since $r' \restriction \max(\vec{s}) \geq r_i \restriction \max(\vec{s}) = \vec{s} = r_j \restriction \max(\vec{s})$, this means that the upper part of r'

The quotient forcing VIII

is incompatible with r_j(which is simply $\langle\kappa, A(r_j)\rangle$), namely, is $\vec{\nu}$ are the ordinals in the part above $\max\vec{s}$ in r' then $\vec{\nu}\notin [A(r_j)]^{<\omega}$. The following generalization of Galvin's theorem [2, P. 143] will suffice to avoid this situation:

Proposition 8

Suppose that $2^{<\kappa}=\kappa$ and let F be a normal filter over κ . Let $\langle X_i \mid i<\kappa^+\rangle$ be a sequence of sets such that for every $i<\kappa^+$, $X_i\in F$, and let $\langle Z_i \mid i<\kappa^+\rangle$ be any sequence of subsets of κ. Then there is $Y\subseteq\kappa^+$ of cardinality κ, and $\alpha^*\in\kappa^+\setminus Y$ such that

1. $\bigcap_{i\in Y} X_i\in F$.
2. $[Z_{\alpha^*}]^{<\omega}\subseteq\bigcup_{i\in Y}[Z_i]^{<\omega}$.

Apply lemma 8 to $X_i=A(q_i)$, $F=\cap_{\xi<o^{\vec{U}}(\kappa)}U(\kappa,\xi)$ and $Z_i=A(r_i)$. There is $Y\subseteq X$ of cardinality κ, and $\alpha^*\in X\setminus Y$ such that

1. $\bigcap_{i\in Y} A(q_i)\in\bigcap_{i<\kappa} U(\kappa,i)$.
2. $[A(r_{\alpha^*})]^{<\omega}\subseteq\bigcup_{i\in Y}[A(r_i)]^{<\omega}$

The quotient forcing IX

Consider the set $A = A(q_{\alpha^*}) \cap (\bigcap_{i\in Y} A(q_i))$. For every $i \in Y$, $q_i \leq \vec{t}^\frown\langle\kappa, A\rangle =: q^*$. Then by $(*)_{\alpha^*}$, there is $r'' \geq r_{\alpha^*}$ such that $r'' \Vdash q^* \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$. Hence there $\vec{s} \leq s'' \in \mathbb{M}[\vec{U}] \restriction \max(\kappa(\vec{s}))$, $k < \omega$, $\vec{\nu} \in [A(r_{\alpha^*})]^k$ and $B_1, ..., B_k$ such that

$$r'' = \langle s'', \langle \nu_1, B_1\rangle, ..., \langle \nu_k, B_k\rangle, \langle \kappa, A(r'')\rangle\rangle$$

Since $\vec{\nu} \in [A(r_{\alpha^*})]^{<\omega}$ and by the property of α^*, $\vec{\nu} \in \bigcup_{j\in Y}[A(r_j)]^{<\omega}$. Thus, there is $j \in Y$ such that $\vec{\nu} \in [A(r_j)]^{<\omega}$. Since r_{α^*} and r_j have the same lower part, and $\vec{\nu} \in [A(r_j)]^{<\omega}$, it follows that r'' and r_j are compatible, contradiction.□

Proof of 8: For every $\alpha < \kappa^+$,$\xi < \kappa$ and $\vec{\nu} \in [Z_\alpha]^{<\omega}$, let

$$H_{\alpha,\xi,\vec{\nu}} = \{i < \kappa^+ \mid X_i \cap \xi = X_\alpha \cap \xi \wedge \vec{\nu} \in [Z_i]^{<\omega}\}$$

Note that $\alpha \in H_{\alpha,\xi,\vec{\nu}}$.

Lemma 29

There is $\alpha^* < \kappa^+$ such that for every $\xi < \kappa$ and $\vec{\nu} \in [Z_{\alpha^*}]^{<\omega}$, $|H_{\alpha^*,\xi,\vec{\nu}}| = \kappa^+$

The quotient forcing X

Proof of Lemma. Otherwise, for every $\alpha < \kappa^+$ there is $\xi_\alpha < \kappa$ and $\vec{\nu}_\alpha \in [Z_\alpha]^{<\omega}$ such that $|H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha}| \leq \kappa$. There is $X \subseteq \kappa^+$, $\vec{\nu}^* \in [\kappa]^{<\omega}$ and $\xi^* < \kappa$, such that $|X| = \kappa^+$ and for every

$$\forall \alpha \in X,\ \vec{\nu}_\alpha = \vec{\nu}^* \wedge \xi_\alpha = \xi$$

Since κ is strong limit and $\xi < \kappa$, there are less than κ many possibilities for $X_\alpha \cap \xi^*$. Hence we can shrink X to $X' \subseteq X$ such that $|X'| = \kappa^+$ and find a single set $E^* \subseteq \xi^*$ such that for every $\alpha \in X'$, $X_\alpha \cap \xi^* = E^*$. It follows that for every $\alpha \in X'$:

$$H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha} = H_{\alpha,\xi^*,\vec{\nu}^*} = \{i < \kappa^+ \mid X_i \cap \xi^* = E^* \wedge \vec{\nu}^* \in [Z_i]^{<\omega}\}$$

Hence the set $H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha}$ does not depend on α, which means it is the same for every $\alpha \in X'$. Denote this set by H^*. To see the contradiction, note that for every $\alpha \in X'$, $\alpha \in H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha} = H^*$, thus $X' \subseteq H^*$, hence

$$\kappa^+ = |X'| \leq |H^*| \leq \kappa$$

The quotient forcing XI

contradiction.$\square_{lemma}$

End of proof of proposition 8: Let α^* be as in the claim. Let us define $Y \subseteq \kappa^+$ that witness the lemma. First, enumerate $[Z_{\alpha^*}]^{<\omega}$, $\langle \vec{\nu}_i \mid i < \kappa \rangle$. By recursion, define β_i for $i < \kappa$. At each step we pick $\beta_i \in H_{\alpha^*, i+1, \vec{\nu}_i} \setminus \{\beta_j \mid j < i\}$. It is possible find such β_i, since the cardinality of $H_{\alpha^*, i+1, \vec{\nu}_i}$ is κ^+, and $\{\beta_j \mid j < i\}$ is of size less than κ. Let us prove that $Y = \{\beta_i \mid i < \kappa\}$ is as wanted. Indeed, by definition, it is clear that $|Y| = \kappa$ and also $[Z_{\alpha^*}]^{<\omega} \subseteq \bigcup_{x \in Y} [Z_x]^{<\omega}$.
Let us argue that $\bigcap_{\alpha < \kappa} X_{\beta_\alpha} \in F$. By normality assumption about F,

$$X^* := X_{\alpha^*} \cap \Delta_{i<\kappa} X_{\beta_i} \in F$$

Thus it suffices to prove that $X^* \subseteq \bigcap_{\alpha < \kappa} X_{\beta_\alpha}$. Let $\zeta \in X^*$, then for every $\alpha < \zeta$, $\zeta \in X_{\beta_\alpha}$. For $\alpha \geq \zeta$, recall that $\beta_\alpha \in H_{\alpha^*, \alpha+1, \vec{\nu}_\alpha}$, hence

$$X_{\alpha^*} \cap (\alpha + 1) = X_{\beta_\alpha} \cap (\alpha + 1)$$

and since $\zeta \in X_{\alpha^*} \cap (\alpha + 1)$, $\zeta \in X_{\beta_\alpha}$. We conclude that $\zeta \in \bigcap_{\alpha < \kappa} X_{\beta_i}$, therefore $X^* \subseteq X_{\alpha^*} \cap \bigcap_{\alpha < \kappa} X_{\beta_i}$. $\square$

Outline

Galvin's Property I

Question

Suppose that $P * \underset{\sim}{Q}$ satisfies $\lambda-$c.c.. Let $G * H$ be a generic subset of $P * \underset{\sim}{Q}$. Consider the interpretation Q of $\underset{\sim}{Q}$ in $V[G, H]$. Does it satisfies $\lambda-$c.c.?

Clearly, this is not true in general. The simplest, let P be trivial and Q be the forcing for adding a branch to a Suslin tree. Then, in V^Q, Q will not be c.c.c. anymore. Our attention in theorem 20 is to subforcings and projections of $\mathbb{M}[\vec{U}]$, however, the argument given work for more general Prikry-Type forcings:

Definition 30

Let F be a $\kappa-$complete uniform filter over a set X, for a regular uncountable cardinal κ. We say that F has:

1. The *Galvin property* iff every family of κ^+ members of F has a subfamily of cardinality κ with intersection in F.
2. The *generalized Galvin property* iff it satisfies the conclusion of 8.

Galvin's Property II

Theorem 31

Suppose that $\mathcal{P}$ is either Prikry or Magidor or Magidor-Radin or Radin or Prikry forcing with an ultrafilter satisfying the generalized Galvin Property. Let $\underset{\sim}{Q}$ be a quotient of $\mathcal{P}$ and $G(\mathcal{P})$ be a V-generic subset of $\mathcal{P}$.
Then, the interpretation of $\underset{\sim}{Q}$ in $V[G(\mathcal{P})]$, satisfies $\kappa^+ -$c.c. there.

We do not know how to generalize this theorem to wider classes of Prikry type forcing notions.
For example the following may be the first step:

Question

Is the result valid for a long enough Magidor iteration of the Prikry forcings?

The problem is that there is no single complete enough filter here, and so the Galvin Theorem (or its generalization) does not seem to apply.
The following question looks natural in this context:

Galvin's Property III

Question

Characterize filters (or ultrafilters) which satisfy the Galvin property (or the generalized Galvin property).

Construction by U. Abraham and S. Shelah [1] may be relevant here. They constructed a model in which there is a sequence $\langle C_i \mid i < 2^{\mu^+} \rangle$ in Cub_{μ^+} such that the intersection of any μ^+ clubs in the sequence is of cardinality less that μ. So the filter Cub_{μ^+} does not posses the Galvin property. Additional restrictions here are posed due to S. Garti[8].
The following questions seems to be open:

Question

Assume GCH. Let κ be a regular uncountable cardinal. Is there a κ-complete filter over κ which fails to satisfy the Galvin property?

Let us note that if the ultrafilter is not on κ, then there is such an ultrafilter, namely, a fine κ-complete ultrafilter over $P_\kappa(\kappa^+))$ does not satisfy the Galvin property:

Galvin's Property IV

For every $\alpha < \kappa^+$, let $X_\alpha = \{Z \in P_\kappa(\kappa^+) \mid \alpha \in Z\}$, then $X_\alpha \in U$ since U is fine but the intersection of any κ elements from this sequence of sets is empty.
A fine normal ultrafilter on $P_\kappa(\lambda)$ is used for the supercompact Prikry forcing (see [9] for the definition). Hence, the following question is natural:

Question

Assume GCH and let $\lambda > \kappa$ be a regular cardinal. Is every quotient forcing of the supercompact Prikry forcing also λ^+-c.c. in the generic extension?

Our prime interest is on $\kappa-$complete ultrafilters over a measurable cardinal κ. Note the following:

Proposition 9

It is consistent that every $\kappa-$complete(or even σ-complete) ultrafilter over a measurable cardinal κ has the generalized Galvin property.

This holds in the model $L[U]$, where U is a unique normal measure on κ. In this model every ultrafilter is Rudin-Keisler equivalent to a finite power of U (see for

Galvin's Property V

example [11, Lemma 19.21]. By 35, it is easy to see that all such ultrafilters satisfy the generalized Galvin property. ■

In context of ultrafilters over a measurable, the following is unclear:

Question

Is it consistent to have a κ-complete ultrafilter over κ which does not have the Galvin property?

Question

Is it consistent to have a measurable cardinal κ carrying a $\kappa-$complete ultrafilter which extends the closed unbounded filter Cub_κ (i.e., $Q-$point) which fails to satisfy the Galvin property?

It is possible to produce more examples of ultrafilters (and filters) with generalized Galvin property. The simplest example of this kind will be $U \times W$, where U, W are normal ultrafilters over κ. We will work in a bit more general setting.

Galvin's Property VI

Definition 32

Let F be a uniform $\kappa-$complete filter over a regular uncountable cardinal κ. F is called a *P-point filter* iff there is $\pi : \kappa \to \kappa$ such that

1. π is almost one to one i.e. there is $X \in F$ such that for every $\alpha < \kappa$, $|\pi^{-1}\alpha \cap X| < \kappa$,
2. For every $\{A_i \mid i < \kappa\} \subseteq F$, $\Delta^*_{i<\kappa} A_i = \{\nu < \kappa \mid \forall i < \pi(\nu)(\nu \in A_i)\} \in F$.

Clearly, every normal filter F is a $P-$point, but there are many non-normal $P-$points as well. For example take a normal filter U and move it to a non-normal by using a permutation on κ. Also, if F is an ultrafilter, then π is just a function representing κ in the ultrapower by F.

Definition 33

Let $F_1, ..., F_n$ be P-point filters over κ, and let $\pi_1, ..., \pi_n$ be the witnessing functions for it. Denote by $[\kappa]^{n*}$, the set of all n-tuples $\langle \alpha_1, .., \alpha_n \rangle$ such that for every $2 \le i \le n$, $\alpha_{i-1} < \pi_i(\alpha_i)$.

Galvin's Property VII

Note that if F_i's are normal, the $\pi_i = id$ and $[\kappa]^{n*} = [\kappa]^n$.

Definition 34

Let $F_1, ..., F_n$ be P-point filters over κ, and let $\pi_1, ..., \pi_n$ be the witnessing functions for it. Define a filter $\prod_{i=1}^{n*} F_i$ over $[\kappa]^{n*}$ recursively. For $X \subseteq [\kappa]^{n*}$:

$$X \in \prod_{i=1}^{n*} F_i \Leftrightarrow \left\{\alpha < \kappa \mid X_\alpha \in \prod_{i=2}^{n*} F_i\right\} \in F_1$$

Where $X_\alpha = \{\langle \alpha_2, ..., \alpha_n \rangle \in [\kappa]^{n-1*} \mid \langle \alpha, \alpha_2, ..., \alpha_n \rangle \in X\}$.

Again, if the filters are normal, this is simply a product.

Proposition 10

Let $F_1, ..., F_n$ be P-point filters over κ, and let $\pi_1, ..., \pi_n$ be the witnessing functions for it. Then for every $X \in \prod_{i=1}^{n*} F_i$, there are $X_i \in F_i$ such that $\prod_{i=1}^{n*} X_i \subseteq X$.

Galvin's Property VIII

By induction on n, for $n = 1$, it is clear. Let $X \in \prod_{i=1}^{n*} F_i$. Let

$$X_1 = \left\{\alpha < \kappa \mid X_\alpha \in \prod_{i=2}^{n*} F_i\right\} \in F_1$$

For every $\alpha \in X_1$, find by induction hypothesis $X_{\alpha,i} \in F_i$ for $2 \leq i \leq n$ such that $\prod_{i=2}^{n*} X_{\alpha,i} \subseteq X_\alpha$. Define

$$X_i = \Delta^*_{\alpha<\kappa} X_{\alpha,i}$$

since F_i is P-point, $X_i \in F_i$. Let us argue that $\prod_{i=1}^{n*} X_i \subseteq X$. Let $\langle \alpha_1, .., \alpha_n \rangle \in \prod_{i=1}^{n*} X_i$, then for every $2 \leq i \leq n$, $\alpha_1 < \pi(\alpha_i)$, hence $\alpha_i \in X_{\alpha_1,i}$. It follows that $\langle \alpha_2, ..., \alpha_n \rangle \in \prod_{i=2}^{n*} X_{\alpha_1,i} \subseteq X_{\alpha_1}$. By definition of X_{α_1}, $\langle \alpha_1, \alpha_2 ... \alpha_n \rangle \in X$.■

Corollary 35

Let $F_1, ..., F_n$ be P-point filters over κ, and let $\pi_1, ..., \pi_n$ be the witnessing functions for it. Then $\prod_{i=1}^{n*} F_i$ also satisfy the generalized Galvin property of 8.

Galvin's Property IX

Let $\langle X_i \mid i < \kappa^+\rangle$ and $\langle Z_i \mid i < \kappa^+\rangle$ as in 8. By proposition 10, for every $1 \leq j \leq n$, and $i < \kappa^+$, find $X_j^{(i)} \in F_j$ such that $\prod_{j=1}^{*n} X_j^{(i)} \subseteq X_i$.
For every $\vec{\xi} = \langle \xi_1, ..., \xi_n\rangle \in [\kappa]^{*n}$ every $\vec{\nu} \in [\kappa]^{<\omega}$ and every $\alpha < \kappa^+$, define

$$H_{\alpha,\vec{\xi},\vec{\nu}} = \left\{\gamma < \kappa^+ \mid \forall 1 \leq i \leq n. X_i^{(\gamma)} \cap \xi_i = X_i^{(\alpha)} \cap \xi_i \text{ and } \vec{\nu} \in [Z_\gamma]^{<\omega}\right\}$$

As in 8, for a fix $\vec{\xi}$, there are less than κ many possibilities for $\langle X_1^{(\alpha)} \cap \xi_1, X_2^{(\alpha)} \cap \xi_2, ..., X_n^{(\alpha)} \cap \xi_n\rangle$, hence we can find $\alpha^* < \kappa^+$, such that for every $\vec{\xi}$ and $\vec{\nu}$, $|H_{\alpha^*,\vec{\xi},\vec{\nu}}| = \kappa^+$.
Enumerate $[Z_{\alpha^*}]^{<\omega}$ by $\langle \vec{\nu}_i \mid i < \kappa\rangle$ and also each F_i is P-point, so for every $j < \kappa$, there is $\rho_i^{(j)} > \sup(\pi_i^{-1''}[j] \cap B_i)$ for some set $B_i \in F_i$. Define the sequence β_j by induction,

$$\beta_j \in H_{\alpha^*,\langle \rho_1^{(j)},...,\rho_n^{(j)}\rangle,\vec{\nu}_j} \setminus \{\beta_k \mid k < j\}$$

We claim once again that

$$X_{\alpha^*} \cap \bigcap_{j<\kappa} X_{\beta_j} \in \prod_{i=1}^{n*} F_i$$

Galvin's Property X

To see this, define for every $1 \leq i \leq n$

$$C_i := X_i^{(\alpha^*)} \cap \Delta^*_{j<\kappa} X_i^{(\beta_j)} \in F_i$$

Let $\vec{\alpha} \in \prod_{i=1}^{n*} C_i$, and let $j < \kappa$. For every $1 \leq i \leq n$, if $j < \pi(\alpha_i)$ then $\alpha_i \in X_i^{(\beta_j)}$. If $\pi(\alpha_i) \leq j$, then $\alpha_i < \rho_i^{(j)}$, so $\alpha_i \in X^{(\alpha^*)} \cap \rho_i^{(j)}$. Since $\beta_j \in H_{\alpha^*, \langle \rho_1^{(j)}, \ldots, \rho_n^{(j)} \rangle, \vec{\nu}_j}$,

$$\alpha_i \in X^{(\alpha^*)} \cap \rho_i^{(j)} = X^{(\beta_j)} \cap \rho_i^{(j)}$$

Therefore, $\vec{\alpha} \in \prod_{i=1}^{n*} X_i^{(\beta_j)} \subseteq X_{\beta_j}$. The continuation is as in 8.■

Outline

The Tree-Prikry forcing I

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Finish line

Thank you for your attention!