Models of Set Theory: Extensions and Dead-ends

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Basics (1)

- Let $\mathsf{ZF}^{-\infty}$ be $\mathsf{ZF} \setminus \{\text{Infinity}\} \cup \{\neg\text{Infinity}\} + \mathsf{TC}$ (where TC asserts that the transitive closure of every set exists). It is well-known that $\mathsf{ZF}^{-\infty}$ and PA are essentially the same theory. They are definitionally equivalent (aka synonymous), i.e., they have a common definitional extension, a notion that is stronger than bi-interpretability.
- We will use $\mathsf{ZF}^{\pm\infty}$ to simultaneously refer to ZF and $\mathsf{ZF}^{-\infty}$.
- We say "$\mathcal{M}$ is a model of set theory" if $\mathcal{M} \models \mathsf{ZF}^{\pm\infty}$. $\mathcal{M} = (M, E)$, where $E \subseteq M^2$ and $E = \in^{\mathcal{M}}$. So $\mathcal{M}$ is a directed graph with certain properties.
- $\mathcal{M} = (M, E)$ is a *standard* model if E is well-founded (in the "real world"). By Mostowski's Collapse, $\mathcal{M}$ is well-founded iff $\mathcal{M}$ is isomorphic to a model of the form $(X, \in)$ where $\in$ is the "real" membership relation.
- Up to isomorphism, the only standard model of $\mathsf{ZF}^{-\infty}$ is $(V_\omega, \in)$.
- Even ZF + Con(ZF) cannot prove that ZF has a standard model. But if ZF has at least one standard model, it has continuum-many nonisomorphic standard models (using forcing).
- $\mathcal{M} = (M, E)$ is nonstandard if it is not standard, i.e. E is ill-founded. By compactness, every standard model has a nonstandard elementary extension.

Basics (2)

- Given a model $\mathcal{M}$ of set theory, we use $\mathrm{Ord}^{\mathcal{M}}$ to denote the "ordinals" of $\mathcal{M}$, i.e., the set of $m \in \mathcal{M}$ such that $\mathcal{M}$ satisfies the formula expressing m is transitive and $\in$ is a (strict) linear order on the elements of m.
- More generally, for a definition φ of an object within ZF, we write $\varphi^{\mathcal{M}}$ for the object in $\mathcal{M}$ satisfying φ.
- So we write $\omega^{\mathcal{M}}$ to denote the ω of $\mathcal{M}$ (i.e., the first nonzero limit ordinal in the sense of $\mathcal{M}$).
- And we write $\mathrm{V}_{\alpha}^{\mathcal{M}}$ for the element a of M such that $\mathcal{M} \models a = \mathrm{V}_{\alpha}^{\mathcal{M}}$ (where $\alpha \in \mathrm{Ord}^{\mathcal{M}}$).
- **Proposition.** *Suppose $\mathcal{M} = (M, E)$ is a model of set theory. $\mathcal{M}$ is well-founded iff $(\mathrm{Ord}^{\mathcal{M}}, E)$ is well-founded.*

Basics (3)

Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\text{set}}$-structures, and $\mathcal{M} \subseteq \mathcal{N}$ (in the sense of model theory, i.e., $\in^{\mathcal{M}} = \in^{\mathcal{N}} \cap M^2$). Let $E = \in^{\mathcal{M}}$.

- For $c \in M$, $\mathrm{Ext}_{\mathcal{M}}(c) := \{m \in M : mEc\} = \{m \in M : \mathcal{M} \models m \in c\}$.
- Suppose $c \in M$. $\mathcal{N}$ fixes c if $\mathrm{Ext}_{\mathcal{M}}(c) = \mathrm{Ext}_{\mathcal{N}}(c)$, and $\mathcal{N}$ enlarges c if $\mathrm{Ext}_{\mathcal{M}}(c) \subsetneq \mathrm{Ext}_{\mathcal{N}}(c)$.
- Suppose $X \subseteq M$. X is said to be coded in $\mathcal{M}$ if $X = \mathrm{Ext}_{\mathcal{M}}(c)$ for some $c \in M$.
- $\mathcal{M} = (M, E)$ of ZF is ω-standard if $(\mathrm{Ext}_{\mathcal{M}}(\omega^{\mathcal{M}}), E) \cong (\omega, \in)$. If $\mathcal{M}$ is not ω-standard, then we say that $\mathcal{M}$ is ω-nonstandard.
- **Jargon practice.** If $\mathcal{M} = (M, E)$ is an ω-standard model of set theory, and $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, then the externally defined usual Tarskian satisfaction relation for the structure $(\mathrm{Ext}_{\mathcal{M}}(\mathrm{V}_\alpha), E)^{\mathcal{M}}$ is coded in $\mathcal{M}$ (assuming that elements of $\mathrm{Ext}_{\mathcal{M}}(\omega^{\mathcal{M}})$ are identified with the corresponding elements of ω).

The multiverse of models of set theory

- **Gödel-Rosser incompleteness Theorem.** *Every consistent theory T with a computably enumerable set of axioms there is a sentence φ such that neither φ nor $\neg\varphi$ is provable in T.*
- **Corollary.** *There are continuum-many completions of* $\mathsf{ZF}^{\pm\infty}$.
- **Corollary.** $\mathsf{ZF}^{\pm\infty}$ *has continuum-many countable nonstandard nonisomrphic models*.
- **Theorem.** *If T is a consistent extension of* $\mathsf{ZF}^{\pm\infty}$ *and κ is an infinite cardinal, then T has 2^κ nonisomorphic models of power κ.*

Logical types of extensions

Suppose $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{set}}$, and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that $\mathcal{M} \subseteq \mathcal{N}$.

(a) For a subset Γ of $\mathcal{L}$-formulae, $\mathcal{M}$ is a Γ-elementary submodel of $\mathcal{N}$ (written $\mathcal{M} \preceq_{\Gamma} \mathcal{N}$) if for all n-ary formulae $\gamma \in \Gamma$ and for all $a_1, \cdots, a_n$ in M,

$$\mathcal{M} \models \gamma(a_1, \cdots, a_n) \text{ iff } \mathcal{N} \models \gamma(a_1, \cdots, a_n).$$

(b) $\mathcal{N}$ is a conservative extension of $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{cons}} \mathcal{N}$) if for every $\mathcal{N}$-definable D (parameters allowed), $M \cap D$ is $\mathcal{M}$-definable (parameters allowed).

(c) $\mathcal{N}$ is a minimal elementary extension of $\mathcal{M}$ (written $\mathcal{M} \prec_{\mathrm{min}} \mathcal{N}$) if $\mathcal{M} \prec \mathcal{N}$ and there is no model $\mathcal{K}$ such that $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$.

Geometrical types of Extensions

Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of set theory, with $\mathcal{M} \subseteq \mathcal{N}$.

old = elements of M, new = elements of $N \setminus M$.

(a) $\mathcal{M}$ is cofinal in $\mathcal{N}$ (written $\mathcal{M} \subseteq_{\mathrm{cof}} \mathcal{N}$) if for every $b \in N$ there is some $a \in M$ such that $b \in \mathrm{Ext}_{\mathcal{N}}(a)$. Every new element is a member of an old element.

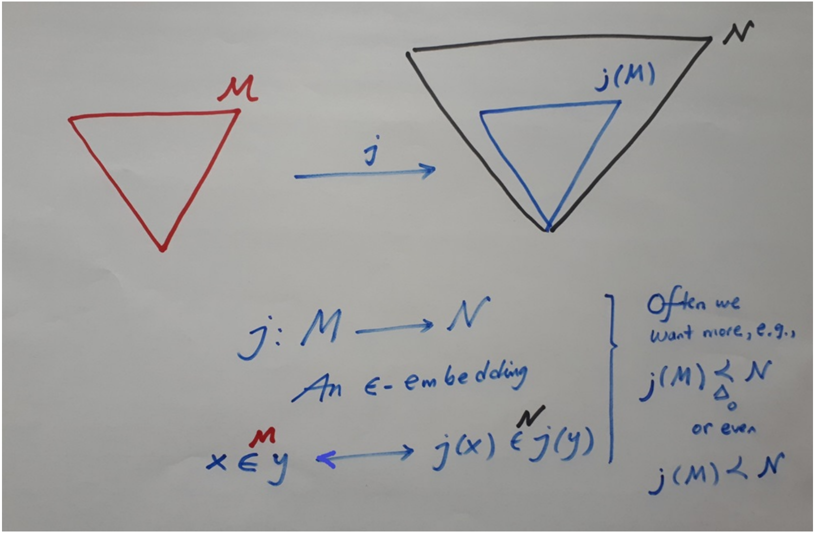
(b) $\mathcal{M}^*$ *is the* convex hull of $\mathcal{M}$ *in* $\mathcal{N}$ if $M^* = \bigcup_{a \in M} \mathrm{Ext}_{\mathcal{N}}(a)$. So M^* is the collection of elements of N that are members of old elements.

(c) $\mathcal{N}$ end extends $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{end}} \mathcal{N}$) if $\mathcal{N}$ fixes every $a \in M$. End extensions are also referred to in the literature as *transitive* extensions, and in the old days as *outer* extensions. Old elements do not gain new members.

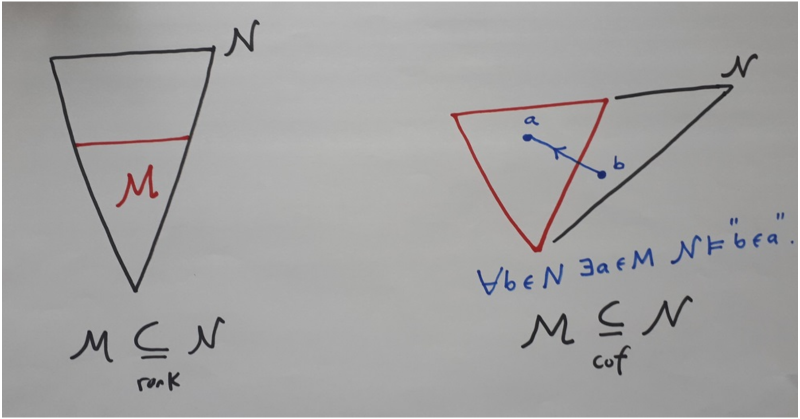
(d) $\mathcal{N}$ is a rank extension of $\mathcal{N}$ (written $\mathcal{M} \subseteq_{\mathrm{rank}} \mathcal{N}$) if for all $a \in M$ and all $b \in N \backslash M$, $\mathcal{N} \models \mathrm{rank}(a) \in \mathrm{rank}(b)$. The rank of every new element is greater than the rank of every old element.

(e) $\mathcal{N}$ is taller than $\mathcal{M}$ if there is some b such that $M \subseteq \mathrm{Ext}_{\mathcal{N}}(b)$. This is equivalent to the existence of an "ordinal" in $\mathcal{N}$ that exceeds all the "ordinals" in $\mathcal{M}$.

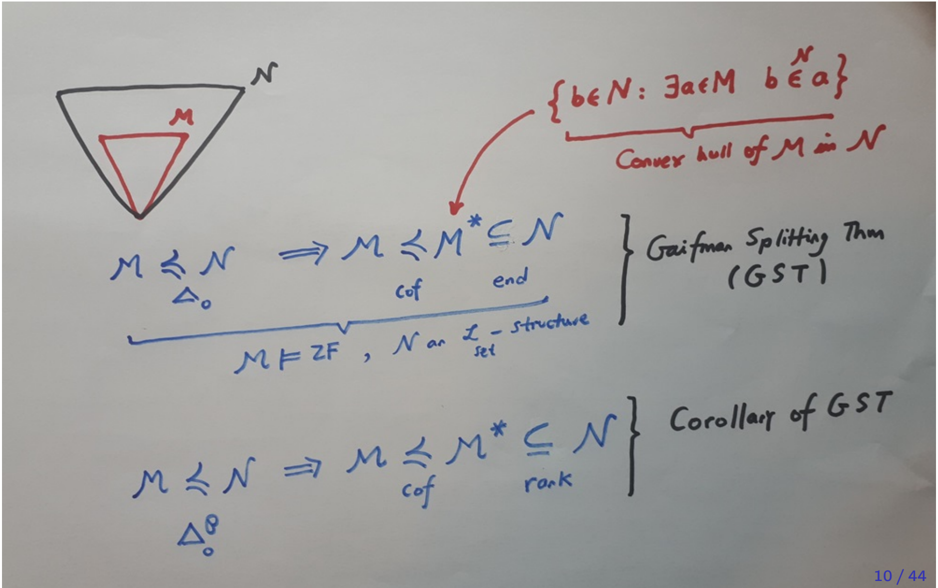
Picture (an embedding)



Picture (rank vs cof)



Picture (Gaifman splitting)



Basic examples

1. Using a simple compactness argument, every model $\mathcal{M}$ can be elementarily extended to a model $\mathcal{N}$ such that $\mathcal{N}$ is taller than $\mathcal{M}$.
2. If $\mathcal{M} \models$ ZFC (of any cardinality), and $\mathcal{U}$ is a nonprincipal ultrafilter in the sense of $\mathcal{M}$, and $\mathcal{N} =$ the internal ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$, then:

$$\mathcal{M} \prec_{\text{cons, cof}} \mathcal{N}.$$

3. If $\mathcal{U}$ is κ-complete in the sense of $\mathcal{M}$ (for some cardinal κ of $\mathcal{M}$), then $\mathcal{N}$ fixes every element in $\mathrm{Ext}_{\mathcal{M}}(\mathrm{V}_{\kappa}^{\mathcal{M}})$, and $\mathcal{N}$ enlarges κ.
4. If $\mathcal{U}$ is a Rudin-Keisler minimal ultrafilter, then additionally:

$$\mathcal{M} \preceq_{\text{min}} \mathcal{N}.$$

5. If κ is a measurable cardinal of $\mathcal{M}$, using a normal $\mathcal{U}$ we get:

$$\mathcal{M} \prec_{\text{cons, cof, min}} \mathcal{N}.$$

Elementary end extensions of models of $\mathrm{ZF}^{-\infty}$

- **Theorem** (MacDowell-Specker 1959, Gaifman and Phillips early 1970s)

  (a) ***Every*** *model* $\mathcal{M}$ *of* PA (equivalently: of $\mathrm{ZF}^{-\infty}$) *has a proper e.e.e.* $\mathcal{N}$.
  (e.e.e. = elementary end extension)

  (b) *Moreover,* $\mathcal{N}$ *can be required to be a* ***minimal and conservative*** *extension of* $\mathcal{M}$.

  Corollary. *Every model of* $\mathrm{ZF}^{-\infty}$ *has an arbitrarily large* κ*-like e.e.e.* (i.e., an e.e.e. of size κ each proper initial segment of which is of size less than κ).
- For set theorists: The proof of the MacDowell-Specker uses a definable ultrapower construction modulo an "iterable ultrafilter".
- In the context of set theory, ultrapowers were first used in the celebrated 1961 proof of Scott that shows that the existence of a measurable cardinal contradicts the axiom $V = L$, at about the same time they were used by Keisler in the investigation of weakly compact cardinals (Keisler proved that κ is weakly compact iff for all $X \subseteq \mathrm{V}_\kappa$, the model $(\mathrm{V}_\kappa, \in, X)$ has an e.e.e.).
- For model-theorists: Gaifman reformulated the MacDowell-Specker in terms of "definable types" and their iterations.

Historical Break: Some Quotes

- Finite iterations of ultrapowers were developed by Frayne, Morel, and Scott. The infinite iterations were introduced by Gaifman. Our presentation is a simplification of Gaifman's work. Independently, Kunen developed iterated ultrapowers in essentially the same way as here, and generalized the construction even further to study models of set theory and measurable cardinals. Chang and Keisler, (Model Theory)
- Some trustworthy witnesses assert that the notion of definable types was not introduced in 1968 by Shelah, but by Haim Gaifman, in order to construct end extensions of models of arithmetic. Poizat (Course in Model Theory)
- There is a paradoxical link between Gaifman's paper and stable first-order theories: although the notion of definable type was introduced by Gaifman in the study of PA, which is the most unstable theory, this notion turned out to be a fundamental one for stable theories. Ressayre (JSL)

Gaifman Splitting for models of ZF

- **Gaifman Splitting Theorem.** *Suppose $\mathcal{M} \models \mathrm{ZF}$, $\mathcal{N}$ is an $\mathcal{L}_{\mathrm{set}}$-structure with $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$, and $\mathcal{M}^*$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$. Then the following hold*:

  (a) $\mathcal{M} \preceq_{\mathrm{cof}} \mathcal{M}^* \preceq_{\Delta_0,\mathrm{end}} \mathcal{N}$.

  (b) *If $\mathcal{M} \prec \mathcal{N}$, then $\mathcal{M} \preceq_{\mathrm{cof}} \mathcal{M}^* \preceq_{\mathrm{end}} \mathcal{N}$.*

- **Definition and Remark.**

  1. $\mathcal{N}$ is a powerset-preserving end extension of $\mathcal{M}$, written $\mathcal{M} \subseteq^{\mathcal{P}}_{\mathrm{end}}$, if $\mathcal{N}$ is an end extension of $\mathcal{M}$ that does not introduce any new subsets of any $a \in \mathcal{M}$
  2. For models $\mathcal{M}$ and $\mathcal{N}$ of ZF we have:

$$\mathcal{M} \preceq_{\Sigma_1,\mathrm{end}} \mathcal{N} \Longrightarrow \mathcal{M} \subseteq^{\mathcal{P}}_{\mathrm{end}} \mathcal{N} \Longrightarrow \mathcal{M} \subseteq_{\mathrm{rank}} \mathcal{N}.$$

Elementary end extensions of models of ZF

- **Theorem** (Keisler 1966). *Every countable model of* $\mathsf{ZF}^{\pm\infty}$ *has an e.e.e.*
- The proof of the the above theorem uses the Omitting Types Theorem of model theory (and takes advantage of the fact that the Collection Scheme is provable in $\mathsf{ZF}^{\pm\infty}$).
- **Corollary.** *Every countable model of* $\mathsf{ZF}^{\pm\infty}$ *has a proper e.e.e. that is* ω_1*-like (i.e., it is uncountable but every proper rank initial submodel of it is countable).*
- $\mathcal{M}$ has countable cofinality if there is an ω-sequence in $\mathrm{Ord}^{\mathcal{M}}$ that is unbounded in $\mathrm{Ord}^{\mathcal{M}}$.
- **Theorem.** (Keisler-Morley 1968) *Every model* $\mathcal{M}$ *of* ZF *that has countable cofinality has a proper e.e.e. of any prescribed cardinality.*
- The proof of the above theorem uses the technology of indiscernibles, as well as the Erdős-Rado Partition Theorem. The proof uses similar ideas as in the proof of a theorem of Morley that states that if an $\mathcal{L}_{\omega_1,\omega}$ sentence has a model of power $\beth_\alpha$ for each $\alpha < \omega_1$, then φ has arbitrarily large models.

The analogy breaks

- **Theorem.** (Keisler-Silver early 1970s) *If κ is the first inaccessible cardinal, then $(V_\kappa, \in)$ has no proper e.e.e.*
- **Remark.** An e.e.e. of a model of ZF is a rank extension.
- **Theorem** (Kaufmann and E. mid 1980s)
  - (a) *No model of* ZFC *has a conservative proper e.e.e.*
  - (b) *Every consistent extension of* ZFC *has a model $\mathcal{M}$ of power $\aleph_1$ such that $\mathcal{M}$ has no e.e.e.*

  Remark. The above theorem remains true if ZF is replaced with ZF($\mathcal{L}$) for finite $\mathcal{L} \supseteq \mathcal{L}_{\text{set}}$, and "e.e.e." is modified to $\mathcal{L}$-elementary end extension.

  Remark. As we shall see in the second part of the talk, ZFC can be weakened to ZF in the above Theorem.

A combinatorial explanation

- **Theorem.** (Essentially König 1927) $\mathsf{ZF}^{-\infty}$ *can prove that every definable subtree of the binary tree of height* Ord *has a definable unbounded branch* (*as a scheme*).

- **Theorem.** (E. 2001, E. and Hamkins 2018) *There is a* $\mathcal{L}_{\mathsf{ZF}}$*-formula* τ *that, provably in* ZFC *defines a subtree of the binary tree of height* Ord *with the property: NO definable branch of* $\tau^{\mathcal{M}}$ *that has height Ord is definable in* $\mathcal{M}$*.*

  Thus ZFC proves that Ord is not definably weakly compact.

How to help ZFC catch up with $\mathrm{ZF}^{-\infty}$

- For an ordinal α, the *α-Mahlo cardinals* are defined recursively as follows:
  1. κ is 0-Mahlo if κ is strongly inaccessible;
  2. For $\alpha = \delta + 1$, κ is α-Mahlo if $\{\gamma < \kappa : \gamma$ is δ-Mahlo$\}$ is stationary in κ;
  3. For limit α, κ is α-Mahlo if κ is δ-Mahlo for all $\delta < \alpha$.
- **Levy Scheme**: $\Lambda := \{(\exists\kappa(\kappa \text{ is } n\text{−Mahlo and } V_\kappa \prec_{\Sigma_n} V) : n \in \omega\}$.
- **Theorem** (Kaufmann and E. late 1980s) *The following are equivalent for a completion of T of* ZFC:

  (a) *There is a consistent extension T^* of T in a countable language $\mathcal{L}^+ \supseteq \mathcal{L}_{\mathsf{ZF}}$ such that* $\mathrm{ZFC}(\mathcal{L}^+) \subseteq T^*$ *and every model of T^* has a proper conservative e.e.e.*

  (b) Λ *is provable in T.*

- **SLOGAN:** ZFC $+ \Lambda$ is the weakest extension of ZFC that allows infinite set theory to model-theoretically imitate finite set theory! The following theorem goes far in explaining this phenomenon.
- **Theorem** (E. 2004, 2022) ZFC $+ \Lambda$ *is precisely what* GBC + "Ord is weakly compact" *knows about* V.

Main results of my recent arXiv paper

- **Theorem A**. *Every model* $\mathcal{M}$ *of* ZF$+\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ *has a conservative elementary extension* $\mathcal{N}$ *that contains an ordinal above all of the ordinals of* $\mathcal{M}$. This theorem is an analogue of the MacDowell-Specker theorem for set theory.
- **Theorem B**. *If* $\mathcal{N}$ *is a conservative elementary extension of a model* $\mathcal{M}$ *of* ZFC, *and* $\mathcal{N}$ *has the same natural numbers as* $\mathcal{M}$, *then* $\mathcal{M}$ *is cofinal in* $\mathcal{N}$.

  In contrast, for models of $\mathrm{ZF}^{-\infty}$, conservative elementary extensions are always end extensions
- **Theorem C.** *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* ZF, *and* $\mathcal{M} \subsetneq_{\mathrm{end,faithful}} \mathcal{N}$. *Then there is some* $\gamma \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ *such that* $\mathcal{M} \preceq \mathcal{N}_\gamma$. *Thus either* $\mathcal{N}$ *is a topped rank extension of* $\mathcal{M}$, *or* $\mathcal{M} \prec \mathcal{N}_\gamma$.

  $\mathcal{N}$ is a *faithful* extension of $\mathcal{M}$ if for every $\mathcal{N}$-definable D, $M \cap D$ is $\mathcal{M}$-amenable, i.e. $(M, M \cap D)$ satisfies ZF in the extended language. This theorem plays a key role in the proof (obtained in collaboration with Mateusz Łełyk) that there is a schematic presentation of ZF that is strongly internally categorical (in constrast with the usual axiomatization of ZF).
- **Theorem D**. *Every consistent extension of* ZF *has a model* $\mathcal{M}$ *of power* $\aleph_1$ *such that* $\mathcal{M}$ *has no proper end extension to a model of* ZF.

  This theorem answers a question posed in my 1984 doctoral thesis. Back then, it had just been shown that the above holds with "end extension" strenghened to "rank extension".

Some References

1. H. J. Keisler and M. Morley, *Elementary extensions of models of set theory*, **Israel J. Math.** (1968).
2. H. Gaifman, *Elementary embeddings of models of set-theory and certain subtheories*, **Axiomatic set theory , Part II**, 1974. American Mathematical Society, Providence, RI, 1974
3. M. Kaufmann, *Blunt and topless end extensions of models of set theory*, **Journal of Symbolic Logic** (1983).
4. A. Enayat, *Analogues of the MacDowell-Specker theorem in set theory*, in **Models, Algebras and Proofs**, edited by X. Caicedo and C.H. Montenegro, Marcel Dekker Inc., 1998.
5. A. Enayat and J. D. Hamkins, ZFC *proves that* Ord *is not weakly compact for definable classes*, **Journal of Symbolic Logic** (2018).
6. A. Enayat, M. Kaufmann, and Z. McKenzie, *Iterated ultrapowers for the masses*, **Archive for Mathematical Logic** (2018).
7. A. Enayat, *Set theory with a proper class of indiscernibles*, **Fundamenta Mathematicae** (2022).
8. A. Enayat and M. Łełyk, *Categoricity-like properties in the first-order realm*, **Journal of Philosophy of Mathematics** (2024).
9. A. Enayat, *Models of set theory: extensions and dead-ends* (2024), preprint: http://arxiv.org/abs/2406.14790

End of Part 1

Warm-up: building e.e.e.s with ultrapowers (1)

Theorem. *Every countable model $\mathcal{M}$ of* $\mathrm{ZF}^{-\infty}$, or of ZFC *has a proper e.e.e.*

Proof: Models of $\mathrm{ZF}^{-\infty}$ carry a definable well-ordering. In case $\mathcal{M}$ is a model of ZFC that has no definable global well ordering, we can work with $(\mathcal{M}, \triangleleft)$, where $\triangleleft$ is a generic global well-ordering.

Let $\mathbb{B}$ be the Boolean algebra of all $\mathcal{M}$-definable subsets of M, $\mathcal{F}$ be the collection of all $\mathcal{M}$-definable functions from M to M, and $\mathcal{F}_0$ be the collection of all $f \in \mathcal{F}$ such that the range of f is bounded (and therefore coded) in $\mathcal{M}$.

Fix an enumeration $\langle f_n : n \in \omega \rangle$ of $\mathcal{F}_0$. By a simple recursion one can construct $S_0 \supseteq S_1 \supseteq \cdots$ such that $S_n \in \mathbb{B}$, S_n is unbounded in M, and f_n is constant on S_n.

Let $\mathcal{U}_0$ consist of elements of $\mathbb{B}$ whose complements are bounded (and therefore coded)in M. It is easy to show that $\{S_n : n \in \omega\} \cup \mathcal{U}_0$ uniquely extends to a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{B}$.

For f and g in $\mathcal{F}$ define: $f \sim g \iff \{m \in M : f(m) = g(m)\} \in \mathcal{U}$.

It is easy to see that $\sim$ is an equivalence relation. Let $M^* := \mathcal{F}/\sim$. So each member of M^* is the $\sim$-equivalence class $[f]$ of some $f \in \mathcal{F}$. For $[f]$, $[g]$, and $[h]$ in M^*, define $\in_{M^*}$ by

$$[f] \in_{M^*} [g] = [h] \iff \{m \in_M M : f(m) \in g(m)\} \in \mathcal{U}.$$

Warm-up: building e.e.e.s with ultrapowers (2)

For each $m \in M$, let $c_m : M \to \{m\}$ be the constant m-function. This defines an embedding j from $\mathcal{M}$ into $\mathcal{M}^*$ given by $m \longmapsto_j [c_m]$.

Łoś-style Theorem. *Suppose $\mathcal{L}_{\mathrm{ZF}}$-formula $\varphi(x_0, \cdots, x_{k-1})$ and $[f_0], \cdots, [f_{k-1}]$ are elements of M^*.* Then we have:

$$\mathcal{M}^* \models \varphi([f_0], \cdots, [f_{k-1}]) \ \leftrightarrow \ \{m \in M : \mathcal{M} \models \varphi(f_0(m), \cdots, f_{k-1}(m))\} \in \mathcal{U}.$$

Proof: Routine induction of the complexity of φ, for the existential step case the global well-ordering is used. □

Therefore the mapping j is an *elementary embedding*, i.e., $j(\mathcal{M}) \preceq \mathcal{M}^*$. Since the equivalence class of the identity function $i(m) = m$ is not in the range of j (since $\mathcal{U}_0 \subseteq \mathcal{U}$), this shows that $\mathcal{M}^* \neq \mathcal{M}$. To see that $\mathcal{M}^*$ *end extends* $\mathcal{M}$, suppose $\mathcal{M}^* \models [f] < [c_{m_0}]$ for some $f \in M^*$ and $m_0 \in M$. Then by the Łoś-style Theorem, we have

$$\overbrace{\{m \in M : \mathcal{M} \models f(m) < m_0\}}^{X} \in \mathcal{U}.$$

Let $f'(m) := f(m)$ if $m \in X$, and otherwise $f'(m) := 0$. Note that $[f'] = [f]$. Moreover, $f' \in \mathcal{F}_0$ and therefore $f' = f_k$ for some $k \in \omega$, which in turn implies (by design) that f' is constant on S_k with some value $m_1 \in M$. So $[f] = [c_{m_1}]$.

An analogue of MacDowell-Specker for ZF (1)

Theorem A. *Every model $\mathcal{M}$ of $\mathrm{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ of any cardinality has a conservative elementary extension $\mathcal{N}$ such that $\mathcal{N}$ is taller than $\mathcal{M}$.*

We shall present two proofs of this theorem:

- The first proof is based on a model-theoretic argument that is reminiscent of the ultrapower proof of the McDowell-Specker theorem. The second proof is short and devilish. The first proof is a bit longer but is more transparent due to its combinatorial flavor. The first proof (but not the first one) yields the following:

  Theorem A^+. *Every model $\mathcal{M}$ of $\mathrm{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ of any cardinality has a conservative elementary extension $\mathcal{N}$ such that $\mathcal{N}$ is taller than $\mathcal{M}$* and we furthermore have:

  $$\mathcal{M} \prec_{\mathrm{cof,\ cons}} \mathcal{M}^* \prec_{\mathrm{end,\ min}} \mathcal{N}.$$

- The second proof is based on a class-sized syntactic construction taking place within a model of set theory; it was inspired by Kaufmann's proof of the MacDowell-Specker theorem using the Arithmetized Completeness Theorem.

The first proof of Theorem A (1)

Given $\mathcal{M} \models \mathsf{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ we can construct an ultrafilter $\mathcal{U}$ over the $\mathcal{M}$-definable subsets of $\mathrm{Ord}^{\mathcal{M}}$ satisfying the following properties:

(1) Every member of $\mathcal{U}$ is unbounded in $\mathrm{Ord}^{\mathcal{M}}$.

(2) Given any $\mathcal{M}$-definable $f : \mathrm{Ord}^{\mathcal{M}} \to \mathcal{M}$, there is a class $X \in \mathcal{U}$ such that either $f \restriction X$ is one-to-one, or the range of $f \restriction X$ is a set (as opposed to a class).

(3) Given any $n \in \omega$, the family $\mathcal{U}_n = \{X \in \mathcal{U} : X \text{ is } \Sigma_n\text{-definable in } \mathcal{M}\}$ is $\mathcal{M}$-definable in the sense that there is a parametric formula $\psi_n(\alpha, \beta)$ such that:

$$X \in \mathcal{U}_n \text{ iff } \exists \alpha \in \mathrm{Ord}^{\mathcal{M}}\ X = \{\beta \in \mathrm{Ord}^{\mathcal{M}} : \mathcal{M} \models \psi_n(\alpha, \beta)\}.$$

We will build $\mathcal{U}$ in $\omega \times \mathrm{Ord}^{\mathcal{M}}$ stages in the following sense: for each $n \in \omega$ and $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ we shall employ *external* recursion on n and an *internal* recursion on α to a parametrically $\mathcal{M}$-definable $X_{n,\alpha} \subseteq \mathrm{Ord}^{\mathcal{M}}$, and then we can define $\mathcal{U}$ as:

$$\mathcal{U} = \{X_{n,\alpha} : n \in \omega, \alpha \in \mathrm{Ord}^{\mathcal{M}}\} \cup \mathcal{U}_{-1},$$

where $\mathcal{U}_{-1}$ is the collection of co-bounded subsets of $\mathrm{Ord}^{\mathcal{M}}$.

The first proof of Theorem A (2)

$\{X_{n,\alpha} : n \in \omega, \alpha \in \mathrm{Ord}^{\mathcal{M}}\}$ can be visualized as the following $\omega \times \mathrm{Ord}^{\mathcal{M}}$ matrix:

$$\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & . & . & . & X_{0,\alpha} & . & . & . \\ X_{1,0} & X_{1,1} & X_{1,2} & . & . & . & X_{1,\alpha} & . & . & . \\ X_{2,0} & X_{2,1} & X_{2,2} & . & . & . & X_{2,\alpha} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ X_{n,0} & X_{n,1} & X_{m,2} & . & . & . & X_{n,\alpha} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

The matrix is constructed row-by-row in ω-steps (external induction), each row is built in $\mathrm{Ord}^{\mathcal{M}}$-steps (internal transfinite induction in the sense of $\mathcal{M}$).

The second proof of Theorem A (1)

We need the following two Facts 1 and 2 below.

FACT 1: *Suppose $\mathcal{M}$ is a model of* ZF *that carries an $\mathcal{M}$-definable global well-ordering*, *and T is an $\mathcal{M}$-definable class of first order sentences such that $\mathcal{M}$ satisfies "T is a consistent first order theory". Then there is a model $\mathcal{N} \models T^{\mathrm{st}}$ such that the elementary diagram of $\mathcal{N}$ is $\mathcal{M}$-definable, where T^{st} is the collection of sentences in $\mathcal{M}$ with standard shape* (i.e., formulae that can be obtained within $\mathcal{M}$ from a standard formula ψ by substituting constants from $\mathcal{M}$ for the free variables of ψ).

Proof of Fact 1. Since $\mathcal{M}$ has a definable global well-ordering, the Henkin proof of the completeness theorem of first order logic can be applied within $\mathcal{M}$ to construct a Henkinized complete extension T^{Henkin} of T (in a language extending the language of T by class-many new constant symbols) such that T^{Henkin} is definable in $\mathcal{M}$. This in turn allows $\mathcal{M}$ to define $\mathcal{N}$ by reading it off T^{Henkin}, as in the usual Henkin proof of the completeness theorem.

The second proof of Theorem A (2)

FACT 2: *If* $\mathcal{M} \models \mathsf{ZF}$, *then for each* $n \in \omega$ $\mathcal{M} \models \mathrm{Con}(\mathrm{Th}_{\Pi_n}(V, \in, c_a)_{a \in \mathrm{V}})$; *here* $\mathrm{Con}(\mathsf{X})$ *expresses the formal consistency of* X, *and* $\mathrm{Th}_{\Pi_n}(V, \in, c_a)_{a \in \mathrm{V}}$ *is the* Π_n*-fragment of the elementary diagram of the universe, which is available in* ZF *by a theorem of Levy*.

Proof of Fact 2. This is an immediate consequence of the Reflection Theorem.

The second proof of Theorem A (3)

Starting with a model $\mathcal{M}$ of $\mathsf{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$, so $\mathcal{M}$ carries a global definable well-ordering. We will construct an increasing sequence of $\mathcal{L}_{\mathrm{set}}$-structures $\langle \mathcal{N}_k : k \in \omega \rangle$ that satisfies the following properties for each $k \in \omega$:

(1) $\mathcal{N}_0 = \mathcal{M}$.

(2) $\mathcal{M} \prec_{\Pi_{k+2}} \mathcal{N}_k \preceq_{\Pi_{k+1}} \mathcal{N}_{k+1}$.

(3) There is an ordinal in $\mathcal{N}_1$ that is above all of the ordinals of $\mathcal{M}$.

(4) $\mathcal{N}_k$ is a conservative extension of $\mathcal{M}$.

Thus

$$\mathcal{M} = \mathcal{N}_0 \preceq_{\Pi_1,\ \mathrm{cons},\ \mathrm{taller}} \mathcal{N}_1 \preceq_{\Pi_2,\ \mathrm{cons}} \mathcal{N}_2 \preceq_{\Pi_3,\ \mathrm{cons}} \mathcal{N}_{3,\ \mathrm{cons}} \cdots \textbf{ and}$$

$$\text{for each } k \in \omega,\ \mathcal{M} \preceq_{\Pi_{n+2},\ \mathrm{cons}} \mathcal{N}_k.$$

This shows that with the choice of $\mathcal{N} := \bigcup_{n \in \omega} \mathcal{N}_n$, we have $\mathcal{M} \prec_{\mathrm{cons}} \mathcal{N}$; and $\mathcal{N}$ is taller than $\mathcal{M}$.

Theorem B^+

- $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}}} \mathcal{N}$ means $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$ and $x = \mathcal{P}(y)$ is absolute between $\mathcal{M}$ and $\mathcal{N}$.
- **Theorem B^+.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are both models of* ZFC *such that $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}},\ \mathrm{faith}} \mathcal{N}$, $\mathcal{N}$ fixes $\omega^{\mathcal{M}}$, and $\mathcal{N}$ is taller than $\mathcal{M}$. Then:*
  - **(a)** *There is some $\gamma \in \mathrm{Ord}^{\mathcal{N}}$ such that $\mathcal{M} \preceq \mathcal{N}_\gamma$.*
  - **(b)** *There is a satisfaction class S for $\mathcal{M}$ such that S can be written as $D \cap M$, where D is $\mathcal{N}$-definable.* By (a) and Tarski's definability of truth.
- **Corollary (Theorem B)**. *A conservative elementary extension of a model of* ZFC *that fixes $\omega^{\mathcal{M}}$ is a cofinal extension.* By (b) above and Tarski's undefinability of truth.

Proof of Theorem B^+ (1)

Fix some $\lambda \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ and some limit $\mathcal{N}$-ordinal $\beta > \lambda$. Within $\mathcal{N}$, thanks to the availability of AC in $\mathcal{N}$, we can let $\lhd$ be a well-ordering of V_β, and then for each $m \in M$, we can define the following set K_m (again within $\mathcal{N}$) as:

$$K_m := \{a \in \mathrm{V}_\beta : \mathcal{N} \models a \in \mathrm{Def}(V_\beta, \in, \lhd, \lambda, m)\},$$

where $x \in \mathrm{Def}(V_\beta, \in, \lhd, \lambda, m)$ is shorthand for the formula of set theory that expresses:

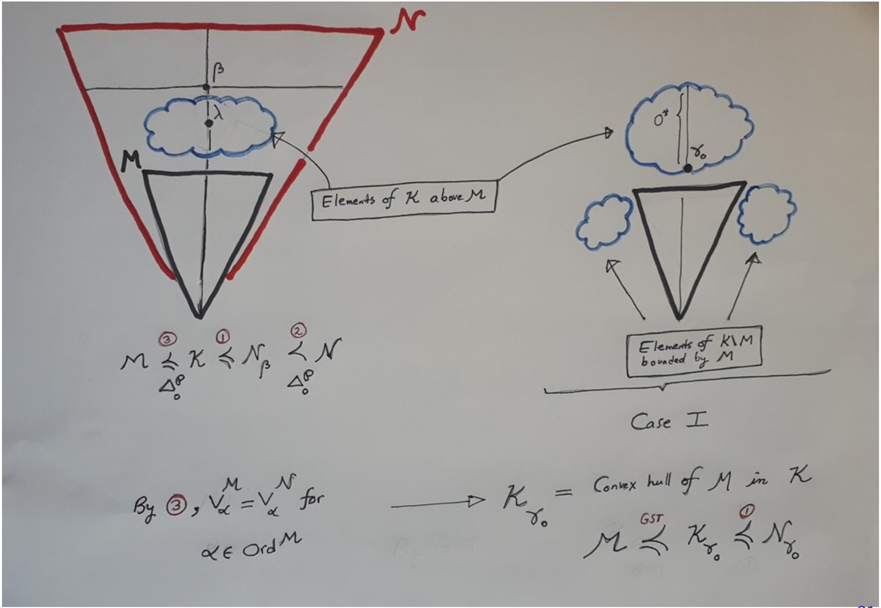
$$x \text{ is definable in the structure } (V_\beta, \in, \lhd, \lambda, m).$$

Thus, intuitively speaking, K_m consists of elements a of N_β such that $\mathcal{N}$ thinks that s is first order definable in $(V_\beta, \in, \lhd, \lambda, m)^{\mathcal{M}}$. Clearly K_m is coded in $\mathcal{N}$. Next we **move outside** of $\mathcal{N}$ and define K as follows:

$$K := \bigcup_{m \in M} K_m.$$

Let $\mathcal{K}$ be the submodel of $\mathcal{N}$ whose universe is K.

Picture of the proof of Theorem B^+



Proof of Theorem B^+ (2)

Using the Tarski test for elementarity (with the help of $\lhd$) we have:

(2) $\mathcal{M} \subsetneq \mathcal{K} \preceq \mathcal{N}_\beta$.

By putting (1) and (2) together we have:

(3) $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}}} \mathcal{K} \preceq \mathcal{N}_\beta \prec_{\Delta_0^{\mathcal{P}}} \mathcal{N}$.

Let O^* be the collection of 'ordinals' of $\mathcal{K}$ that are above the 'ordinals' of $\mathcal{M}$. Clearly O^* is nonempty since $\lambda \in O$. We now consider the following two cases. As we shall see, Case I leads to the conclusion of the theorem, and Case II is impossible.

Case I. O^* has a least ordinal (under $\in^{\mathcal{K}}$).

Case II. O^* has no least ordinal.

Suppose Case I holds and let $\gamma_0 = \min(O^*)$. Clearly γ_0 is a limit ordinal of $\mathcal{N}$. By the choice of γ_0, $\mathrm{Ord}^{\mathcal{M}}$ is cofinal in $\mathrm{Ord}^{\mathcal{K}_{\gamma_0}}$. Since (3) implies that $\mathrm{V}_\alpha^{\mathcal{M}} = \mathrm{V}_\alpha^{\mathcal{K}}$ for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, we conclude:

(4) $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}},\ \mathrm{cof}} \mathcal{K}_{\gamma_0}$.

Proof of Theorem B^+ (3)

Since $\mathcal{K}_{\gamma_0}$ is the convex hull of $\mathcal{M}$ in $\mathcal{K}$, by Gaifman Splitting Theorem, (4) shows that:

(5) $\mathcal{M} \preceq \mathcal{K}_{\gamma_0}$.

On the other hand since by (3) $\mathcal{K} \preceq \mathcal{N}_\beta$, we have:

(6) $\mathcal{K}_{\gamma_0} = \mathrm{V}_{\gamma_0}^{\mathcal{K}} \preceq \mathrm{V}_{\gamma_0}^{\mathcal{N}_\beta} = \mathrm{V}_{\gamma_0}^{\mathcal{N}} = \mathcal{N}_{\gamma_0}$.

By (5) and (6), $\mathcal{M} \preceq \mathcal{N}_{\gamma_0}$, as desired. So Case I leads to the desired conclusion. We will show that Case II is impossible. Within $\mathcal{N}$ let s_β be the Tarskian satisfaction class for $(\mathrm{V}_\beta, \in)$, and let $\Phi := \bigcup_{m \in M} \Phi_m$, where Φ_m is :

$$\{x \in M : \mathcal{N} \models x \text{ is (the code of) a formula } \varphi(c, c_m) \text{ such that } \varphi(c_\lambda, c_m) \in s_\beta\}.$$

So intuitively speaking, Φ is the result of replacing c_λ by c (where c is a fresh constant) in the sentences in the elementary diagram of $\mathcal{N}_\beta$ (as computed in $\mathcal{N}$) whose constants are in $\{c_\lambda\} \cup \{c_m : m \in M\}$. Since $\mathcal{N}$ need not be ω-standard, the elements of Φ might be nonstandard formulae.

Since $\mathcal{N}$ is a faithful extension of $\mathcal{M}$, Φ is $\mathcal{M}$-amenable. Next let
$\Gamma := \{t(c, c_m) \in M : t(c, c_m) \in \Phi \text{ and } \forall \theta \in \mathrm{Ord}\ (t(c, c_m) > c_\theta) \in \Phi\}$,
where t is a *definable term* in the language $\mathcal{L}_{\mathrm{set}} \cup \{c\} \cup \{c_m : m \in M\}$.

Proof of Theorem B^+ (4)

Note that Γ is definable in $(\mathcal{M}, \Phi)$. Since we are considering Case II, $(\mathcal{M}, \Phi) \models \psi$, where:

$$\psi := \forall t \left(t \in \Gamma \longrightarrow [\exists t' \in \Gamma \wedge ((t' \in t) \in \Phi)]\right).$$

The veracity of the dependent choice scheme in $\mathcal{M}$ in models of ZFC (thanks to the Reflection Theorem), together with the facts that Φ is $\mathcal{M}$-amenable and Γ is definable in $(\mathcal{M}, \Phi)$ make it clear that there is a sequence $s = \left\langle t_n : n \in \omega^{\mathcal{M}} \right\rangle$ in $\mathcal{M}$ such that:

$$(\mathcal{M}, \Phi) \models \forall n \in \omega \ \ [t_n \in \Gamma \wedge ((t_{n+1} \in t_n) \in \Phi)].$$

Since s is a countable object in $\mathcal{M}$, and $\mathcal{N}$ fixes ω^M by assumption, s is fixed in the passage from $\mathcal{M}$ to $\mathcal{N}$. On the other hand, since $\mathcal{N}$ has a satisfaction predicate s_β for $\mathcal{N}_\beta$, this leads to a contradiction because we have:

$$\mathcal{N} \models \left\langle t_n^{(\mathrm{V}_\beta, \in)}(c_\lambda) : n \in \omega \right\rangle \text{ is an infinite decreasing sequence of ordinals.}$$

In the above $t_n(c_\lambda)$ is the term obtained by replacing c with c_λ in t_n, and $t_n^{(\mathrm{V}_\beta, \in)}(c_\lambda)$ is the interpretation of $t_n(c_\lambda)$ in $(\mathrm{V}_\beta, \in)$. This concludes the proof of Theorem B^+.

Theorem C and Theorem D^{-}

- **Theorem C.** *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* ZF, *and* $\mathcal{M} \subsetneq_{\mathrm{end,faithful}} \mathcal{N}$. *Then there is some* $\gamma \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ *such that* $\mathcal{M} \preceq \mathcal{N}_{\gamma}$. *Thus either* $\mathcal{N}$ *is a topped rank extension of* $\mathcal{M}$, *or* $\mathcal{M} \prec \mathcal{N}_{\gamma}$.
- Theorem C is established with the same proof strategy as Theorem B^{+}, but due to absence of AC, one needs to work harder.
- **Theorem** D^{-}**.** *No model of* ZF *has a conservative proper end extension satisfying* ZF.
- **Proof.** Put part (b) of Theorem C together with Tarski's Undefinability of Truth Theorem.

Towards Theorem D (1)

Definition. Suppose $\mathcal{M}$ is an $\mathcal{L}_{\text{set}}$-structure.

(a) $X \subseteq M$ is piecewise-coded in $\mathcal{M}$ if
$\forall a \in M\ \exists b \in M\ \mathrm{Ext}_{\mathcal{M}}(b) = X \cap \mathrm{Ext}_{\mathcal{M}}(a)$.

(b) $\mathcal{M}$ is rather classless iff every piecewise-coded subset of M is $\mathcal{M}$-definable.

(c) $\mathcal{M}$ is $\aleph_1$-like if $|M| = \aleph_1$ but $|\mathrm{Ext}_{\mathcal{M}}(a)| \leq \aleph_0$ for each $a \in M$.

Theorem. (Keisler-Kunen 1974, Shelah 1980) *Every countable model of* ZF *has an elementary end extension to an* $\aleph_1$*-like rather classless model*.

Theorem. *No rather classless model of* ZF *has a proper rank extension to a model of* ZF.

Proof. This follows from putting Theorem D^- together with the observation that a rank extension of a model of ZF is a conservative extension. □

Remark. it is possible for a rather classless model to have a proper end extension satisfying ZF, since $\aleph_1$-like rather classless models exist by the above theorem, and one can use the Boolean-valued approach to forcing to construct set generic extensions of such models.

Towards Theorem D (2)

Definition. A ranked tree τ is a two-sorted structure $\tau = (T, \leq_T, L, \leq_L, \rho)$ satisfying the following three properties:

(1) $(T, \leq_T)$ is a tree, i.e., a partial order such that any two elements below a given element are comparable.

(2) $(L, \leq_L)$ is a linear order with no last element.

(3) ρ is an order preserving map from $(T, \leq_T)$ onto $(L, \leq_L)$ with the property that for each $t \in T$, ρ maps the set of predecessors of t onto the initial segment of $(L, \leq_L)$ consisting of elements of L that are less than $\rho(t)$.

Definition. Suppose $\tau = (T, \leq_T, L, \leq_L, \rho)$ is a ranked tree. A linearly ordered subset B of T is said to be a branch of τ if the image of B under ρ is L. The cofinality of τ is the cofinality of $(L, \leq_L)$.

Definition. Given a structure $\mathcal{M}$ in a language $\mathcal{L}$, we say that a ranked tree τ is $\mathcal{M}$-definable if $\tau = \mathrm{t}^{\mathcal{M}}$, where t is an appropriate sequence of $\mathcal{L}$-formulae whose components define the corresponding components of τ in $\mathcal{M}$. $\mathcal{M}$ is rather branchless if for each $\mathcal{M}$-definable ranked tree τ, all branches of τ (if any) are $\mathcal{M}$-definable.

Towards Theorem D (3)

Theorem. *Suppose $\mathcal{M}$ is a countable structure in a countable language.*

(a) (Keisler-Kunen 1974, essentially). *It is a theorem of* $\mathrm{ZFC} + \lozenge_{\omega_1}$ *that $\mathcal{M}$ can be elementarily extended to a rather branchless model*.

(b) (Shelah 1978). *It is a theorem of* ZFC *that $\mathcal{M}$ can be elementarily extended to a rather branchless model*.

Definition. In what follows $(P, \leq_P)$ is a poset.

- $(P, \leq_P)$ is is directed if any pair of given elements of P has a $\leq_P$-upper bound. Clearly every finite subset of a directed set has an upper bound.
- A subset F of $(P, \leq_P)$ is a prefilter over $(P, \leq_P)$ if the sub-poset $(F, \leq_P)$ is directed. F is maximal prefilter over $(P, \leq_P)$, is if it cannot be properly extended to a filter over $(P, \leq_P)$.
- A subset C of $(P, \leq_P)$ is cofinal in $(P, \leq_P)$ if $\forall x \in P\ \exists y \in C\ y \leq_P x$.

Definition. Suppose s is an infinite set.

(a) $[s]^{<\omega}$ is the directed poset of finite subsets of s, ordered by containment.

(b) $\mathrm{Fin}(s, 2)$ is the poset of finite functions from s into $\{0, 1\}$, ordered by containment.

Towards Theorem D (4)

Example. Given an infinite set s, and $a \subseteq s$, let $\chi_a : s \to 2$ be the characteristic function of a, i.e., $\chi_a(x) = 1$ iff $x \in a$. Let F_a be the set of finite approximations to χ_a. F_a is a maximal filter of $\mathrm{Fin}(s, 2)$.

Definition. A structure $\mathcal{M}$ is a Rubin model if it has the following two properties:

(a) Every $\mathcal{M}$-definable directed set with no maximum element has a cofinal chain of length ω_1.

(b) Given any $\mathcal{M}$-definable poset P, and any maximal prefilter $F \subseteq P$, if F has a cofinal chain of length ω_1, then F is coded in $\mathcal{M}$.

Remark. If $\tau = (T,\ \leq_T,\ L,\ \leq_L,\ \rho)$ is a ranked tree, then each branch of τ is a maximal filter over $(T,\ \leq_T)$. This makes it clear that every Rubin model is rather branchless. Also, a rather branchless model of ZF is rather classless. To see this consider the ranked tree defined within a model $\mathcal{M}$ of ZF as follows: The nodes of τ are ordered pairs (s, α), where $s \subseteq \mathrm{V}_\alpha$, the rank of (s, α) is α and $(s, \alpha) < (t, \beta)$ if $\alpha \in \beta$ and $s = t \cap \mathrm{V}_\alpha$. It is easy to see that $\mathcal{M}$ is rather classless iff every branch of $\tau^{\mathcal{M}}$ is $\mathcal{M}$-definable. Hence we have the following chain of implications: Rubin $\Rightarrow$ rather branchless $\Rightarrow$ rather classless.

Towards Theorem D (5)

Theorem. (Rubin 1980). *It is a theorem of* $\mathrm{ZFC} + \diamondsuit_{\omega_1}$ *that if* $\mathcal{M}$ *is a countable structure in a countable language, then* $\mathcal{M}$ *has an elementary extension of cardinality* $\aleph_1$ *that is a Rubin model*.

Definition. Suppose $\mathcal{M}$ is a model of ZF. $\mathcal{M}$ is weakly Rubin if (a) and (b) below hold:

(a) $\mathcal{M}$ is rather classless.

(b) For every element *a* of $\mathcal{M}$ that is infinite in the sense of $\mathcal{M}$ we have:

(i) $([a]^{<\omega})^{\mathcal{M}}$ has a cofinal chain of length ω_1.

(ii) If *F* is a maximal prefilter of $\mathrm{Fin}^{\mathcal{M}}(a, 2)$ and *F* has a cofinal chain of length ω_1, then *F* is coded in $\mathcal{M}$.

Theorem (Rubin-Schmerl) *It is a theorem of* ZFC *that every countable model of* ZF *has an elementary extension to a weakly Rubin model of cardinality* $\aleph_1$.

Using Schmerl's strategy of $\diamondsuit_{\omega_1}$-elimination (2000), an absoluteness theorem of Shelah (1978) can be invoked so as to eliminate the $\diamondsuit_{\omega_1}$ hypothesis in building weakly Rubin models.

Theorem D (1)

Theorem. *Every countable model* $\mathcal{M}_0 \models \mathrm{ZF}$ *has an elementary extension* $\mathcal{M}$ *of power* $\aleph_1$ *that has no proper end extension to a model* $\mathcal{N} \models \mathrm{ZF}$. *Thus every consistent extension of* ZF *has a model of power* $\aleph_1$ *that has no proper end extension to a model of* ZF.

We begin with a basic fact that will be called upon in the proof.

Fact (∇). *Suppose* $\mathcal{M} \models \mathrm{ZF}$ *and* $\mathcal{N} \models \mathrm{ZF}$ *with* $\mathcal{M} \subseteq_{\mathrm{end}} \mathcal{N}$, *and* $a \in M$. *If* $s \in N$ *such that* s *is finite as viewed in* $\mathcal{N}$ *and* $\mathcal{N} \models s \subseteq a$, *then* $s \in M$. *Thus for all* $a \in M$, *we have*:

$$\left([a]^{<\omega}\right)^{\mathcal{M}} = \left([a]^{<\omega}\right)^{\mathcal{N}}.$$

Theorem D (2)

Given a countable model $\mathcal{M}_0$ of ZFC, by the Rubin-Schmerl Theorem there is a weakly Rubin model $\mathcal{M}$ that elementary extends $\mathcal{M}_0$. By Theorem D^-, the proof will be complete once we verify that that every end extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies ZF is a conservative extension.

Towards this goal, suppose $\mathcal{M} \subsetneq_{\mathrm{end}} \mathcal{N} \models$ ZF. The proof will be complete if we could show that $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}}} \mathcal{N}$ since this would assure us in that for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, we have:

$$\mathrm{V}_\alpha^{\mathcal{M}} = \mathrm{V}_\alpha^{\mathcal{N}},$$

The above shows that $\mathcal{N}$ is a proper rank extension of $\mathcal{M}$ since $\mathrm{Ord}^{\mathcal{M}}$ is an initial segment of $\mathrm{Ord}^{\mathcal{N}}$ by the assumption that $\mathcal{N}$ is an end extension of $\mathcal{M}$. But since $\mathcal{M}$ is rather classless, we can deduce that $\mathcal{N}$ is also a conservative rank extension of $\mathcal{M}$ (recall: rank extensions of rather classless models are conservative).

Theorem D (3)

Note that $\mathcal{M} \prec_{\Delta_0} \mathcal{N}$ since $\mathcal{N}$ end extends $\mathcal{M}$, so we just need so show that $x = \mathcal{P}(y)$ is absolute between $\mathcal{M}$ and $\mathcal{N}$. Suppose for $a \in M$ and $s \in N$, $\mathcal{M}$, $\mathcal{N} \models s \subseteq a$ for some $a \in N$. We will show that $s \in M$. By Fact (∇) we may assume that a is infinite in $\mathcal{M}$. Note that this implies that $\mathcal{M}$ views $[a]^{<\omega}$ as a directed set with no maximum element. Also, Fact (∇) assures us that:

$(*)$ $\forall m \in M \ ([m]^{<\omega})^{\mathcal{M}} = ([m]^{<\omega})^{\mathcal{N}}$, and

Since $\mathrm{Fin}(m, 2) \subseteq [m \times \{0, 1\}]^{<\omega}$, $(*)$ implies:

$(**)$ $\forall m \in M \ \mathrm{Fin}^{\mathcal{N}}(m, 2) = \mathrm{Fin}^{\mathcal{M}}(m, 2)$.

Let $\chi_s \in N$ such that $\mathcal{N}$ satisfies $\chi_s : a \to \{0, 1\}$ and $\forall x \in a(x \in s \leftrightarrow \chi_s(x) = 1)$. Let $F_s = ([\chi_s]^{<\omega})^{\mathcal{N}}$.

F_s is a maximal prefilter over $\mathrm{Fin}^{\mathcal{N}}(a, 2)$, so by $(**)$ F_s is a maximal filter over $\mathrm{Fin}^{\mathcal{M}}(a, 2)$. The directed set $([a]^{<\omega})^{\mathcal{M}}$ has a cofinal chain $\langle p_\alpha : \alpha \in \omega_1 \rangle$ thanks to the assumption that $\mathcal{M}$ is weakly Rubin. Together with $(*)$, this readily implies that F_s has a cofinal chain $\langle q_\alpha : \alpha \in \omega_1 \rangle$, where $q_\alpha := \chi_s \restriction p_\alpha$. Therefore, by the assumption that $\mathcal{M}$ is weakly Rubin, F_s is $\mathcal{M}$-definable. In light of the fact that $\chi_s = \cup F_s$, this makes it clear that $s \in N$, which concludes the proof of Theorem D.

Some Open Questions and Gratitude

- If a model $\mathcal{M}$ of ZFC has a taller conservative elementary extension, does $\mathcal{M}$ have to satisfy $\exists p(\mathrm{V} = \mathrm{HOD(p)})$?
- Can ZFC be reduced to ZF in Theorem B?
- Does every consistent extension of ZF that has an ω-standard model have an ω-standard model $\mathcal{M}$ that cannot be properly end extended to a model of ZF?

THANKS FOR YOUR ATTENTION!