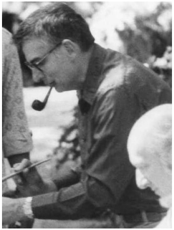
Jim's Doctoral Thesis: A Retrospective

Special Session to Honor Jim Schmerl on His 85th Birthday, MOPA, CUNY

Ali Enayat, April 22, 2025

Jim's thesis (1)

- James H. Schmerl, ***On κ-like models for inaccessible κ***, Doctoral Dissertation, University of California, Berkeley, 1971.
(Advisor: Robert Vaught).



Jim's thesis (2)

- The thesis centers around the notion "κ-like"; and the large cardinal concept (introduced by Levy) "*hyperinaccessible of type n*", nowadays known as *n-Mahlo*.
- Given an infinite cardinal κ, a linearly ordered set $(A, <)$ is κ-like if the cardinality of A is κ but each proper initial segment of $(A, <)$ has cardinality less than κ. This leads to the notion of a κ-like structure.
- As shown by Fuhrken (1964) the model theory of κ-like structures is basically the model theory of first order logic with the counting quantifier Q_κ expressing "there exist κ-many".
- κ is 0-Mahlo means that κ is strongly inaccessible.
- κ is $(\alpha + 1)$-Mahlo means that $\{\lambda < \kappa : \lambda \text{ is } \alpha\text{-Mahlo}\}$ is stationary, i.e., it intersects every closed unbounded subset of κ.
- For limit α, κ is α-Mahlo means that for all $\beta < \alpha$, κ is β-Mahlo.
- 0-Mahlo $<<$ 1-Mahlo $<<$ 2-Mahlo $<<$ ... ω-Mahlo $<<$ $(\omega + 1)$-Mahlo.
- κ is weakly compact $\Longrightarrow$ κ is κ-Mahlo.

Transfer

- For infinite cardinals κ and λ, $\kappa \longrightarrow \lambda$ means: *for every countable theory T with a distinguished ordering $<$, if T has a κ-like model, then T has a λ-like model*. This is equivalent to $\mathrm{Val}_{Q_\lambda} \subseteq \mathrm{Val}_{Q_\kappa}$.
- The question "For which pair of cardinals (κ, λ), does $\kappa \longrightarrow \lambda$ hold?" essentially goes back to Mostowski (1957) in the guise of cardinality quantifiers.
- **Theorem.** (Vaught 1962) $\kappa \longrightarrow \aleph_1$ *for every uncountable regular* κ.
- In 1965 Fuhrken noted that the MacDowell-Specker (1959) theorem (asserting that every model of PA has an elementary end extension) implies:
- **Theorem.** $\aleph_0 \longrightarrow \lambda$ *for all* λ.
- **Theorem.** (Helling 1966) $\kappa \longrightarrow \lambda$, *for weakly compact* κ, *and all uncountable* λ.
- Fuhrken conjectured that in the above theorem, "weakly compact" can be weakened to "strongly inaccessible".
- James H. Schmerl, *Transfer theorems and their applications to logics*. **Model-theoretic logics**, 177-209, Perspect. Math. Logic, Springer, New York, 1985.

Paper on infinite combinatorics (1)

- *A partition property characterizing cardinals hyperinaccessible of finite type*, **Transactions of the American Mathematical Society**, vol. 188 (1974), pp. 281-291.
- The paper introduces the classes $P(n, \alpha)$ of infinite cardinals, where $\kappa \in P(n, \alpha)$ iff the following property holds:

  For each $\gamma < \kappa$, let C_γ be a partition of $[\kappa]^n$ into less than κ parts. Then there is $X \subseteq \kappa$ of order type α such that for each $\gamma \in X$ the set $X \setminus (\gamma + 1)$ is homogeneous for C_γ.

  The various theorems proved in the paper result in the determination of the classes $P(n, \alpha)$ for almost all n and α. Significant results include the following:

  1. $\kappa \in P(n+2, n+4)$ iff κ is a strong limit cardinal;
  2. $\kappa \in P(2, 5)$ iff κ is strongly inaccessible;
  3. $\kappa \in P(n+2, n+5)$ if and only if κ is n-Mahlo; and
  4. $\kappa \in P(n+1, \kappa)$ if and only if κ is weakly compact.

Paper on infinite combinatorics (2)

- András Hajnal, Akihiro Kanamori, and Saharon Shelah, *Regressive Partition Relations for Infinite Cardinals,* **Transactions of the American Mathematical Society**, vol. 299, (1987), pp. 145-154.
- **Abstract of above.** *The regressive partition relation, which turns out to be important in incompleteness phenomena, is completely characterized in the transfinite case. This work is related to Schmerl's work, whose characterizations we complete.*
- For a set X of ordinals, an ordinal γ, and $k \in \omega$, let $X \to (\gamma)^k_{\mathrm{reg}}$ be as in the Erdős-notation for partition calculus, with the stipulation that the colorings be regressive, i.e., satisfy $f\{\alpha_1, ...\alpha_n\} < \alpha_1$ (where $\alpha_1 < ... < \alpha_n$).
- As shown in the above paper, $X \to (\gamma)^{n+3}_{\mathrm{reg}}$ holds iff
  1. EITHER $X \cap \kappa$ is unbounded in κ for some $(n+1)$-Mahlo cardinal $\kappa > \gamma$,
  2. OR ELSE $X \cap \gamma$ is unbounded in γ for some weakly compact γ ($\gamma = \aleph_0$ allowed).

Paper on n-Mahlo cardinals (1)

- James H. Schmerl, *An elementary sentence which has ordered models*, **Journal of Symbolic Logic,** (1972), vol. 37, pp. 521-530.
- The paper develops intricate combinatorial properties of n-Mahlo cardinals, which are then used to establish a general theorem from which the following can be readily derived:
- **Theorem.** *For each $n \in \omega$ there is a first order sentence $\varphi_n(<)$ such that for any regular cardinal κ, $\varphi_n(<)$ has a κ-like model iff κ is not n-Mahlo.*
- In particular, for any fixed n, the condition "κ is n-Mahlo" does NOT imply that for all uncountable λ, $\kappa \longrightarrow \lambda$. This refutes Fuhrken's conjecture.

Paper on n-Mahlo cardinals (2)

- Harvey Friedman, *On the necessary use of abstract set theory*, **Advances in Mathematics** vol. 41 (1981), 209–280.
- Within $\mathrm{ZFC} + \exists\kappa(\kappa \text{ is } \omega\text{-Mahlo})$ Friedman establishes a "Ramsey Borel theorem" and then uses the main result of Jim's 1972 paper to show that there is no $n \in \omega$ for which the Ramsey Borel theorem is provable in $\mathrm{ZFC} + \mathrm{V} = \mathrm{L} + \exists\kappa(\kappa \text{ is } n\text{-Mahlo})$.
- Matt Kaufmann, *Blunt and topless end extensions of models of set theory*, **Journal of Symbolic Logic** (1983).

Matt (Chicago, 1983)

- In the above paper Matt proved a number of striking results about models of set theory. He used one of them to give a "soft" proof of the main result of Jim's paper. Matt communicated this new proof to Jim, and then Matt was informed by Jim that Jack Silver (unpublished) had came up with a similar proof.

Paper on κ-like models (1)

- James H. Schmerl and Saharon Shelah, *On power-like models for hyperinaccessible cardinals*, **The Journal of Symbolic Logic**, vol. 37, (1972), pp. 531-537.
  1. **Theorem.** *IF T is a theory that has an n-Mahlo-like model for each $n < \omega$, THEN for each $\lambda \geq card(T) + \aleph_0$, there is a λ-like model of T.*
  2. The above shows that for all uncountable λ, ω-Mahlo $\longrightarrow \lambda$. This sharpens Helling's theorem (about weakly compact cardinals).
  3. As mentioned earlier, Schmerl's 1972 paper "An elementary sentence which has ordered models" shows that the above is optimal.
  4. As pointed out by the authors of the paper, since the results and methods are correct if $\aleph_0$ is considered as being n-Mahlo for all $n < \omega$, the MacDowell-Specker Theorem can be seen as a special case of the main result of the paper.
  5. The paper also includes an "omitting types" version of the above theorem.

Paper on κ-like models (2)

- **Theorem.** (Kaufmann 1983) *Every consistent extension of* ZFC *has a model of power* $\aleph_1$ *that has no elementary end extension*.
- **Levy Scheme**: $\Lambda := \{\exists\kappa\ (\kappa \text{ is } n\text{--Mahlo and } V_\kappa \prec_{\Sigma_n} V) : n \in \omega\}$.
- **Theorem** (Kaufmann and E. mid-1980s) *The following are equivalent for a completion of* T *of* ZFC:

  (a) T *has a consistent extension* T^* *in a countably infinite language* $\mathcal{L} \supseteq \mathcal{L}_{\mathsf{ZF}}$ *such that* $\mathsf{ZFC}(\mathcal{L}) \subseteq T^*$ *and every model of* T^* *has an elementary end extension.*

  (b) Λ *is provable in* T.
- **SLOGAN:** ZFC $+ \Lambda$ is the weakest extension of ZFC that allows infinite set theory to model-theoretically imitate finite set theory (equivalently: PA)!
- A. E., *Automorphisms, Mahlo cardinals, and NFU*, in **Nonstandard Models of Arithmetic and Set Theory**, Contemporary Mathematics, AMS Publications, 2004.

  ——, *Set theory with a proper class of indiscernibles*, **Fundamenta Mathematicae**, 2022.

Synopsis

- The following elements that are visible in Jim's 1971 dissertation have also been pervasive in his far-ranging mathematical work ever since:
  1. Identification of the combinatorial core of model-theoretic phenomena.

     James H. Schmerl, *Kernels, truth and satisfaction*, **Bull. Pol. Acad. Sci. Math.** (2019).

  2. Development of the model theory of generalized quantifiers (including Ramsey quantifiers, hyper-Ramsey quantifiers, and the stationary quantifier).

     James, H. Schmerl, *PA(aa)*, **Notre Dame J. Formal Logic** (1995).

  3. Formulation of various extensions and elaborations of the MacDowell-Specker theorem, and their application in unexpected settings, e.g., in the complete solution to the "Sikorski problem" concerning ordered fields.

     James H. Schmerl, *Models of Peano arithmetic and a question of Sikorski on ordered fields*, **Israel J. Math.** (1985).

Thanks Jim for "showing the way"

Jim, Roman, and Ali (2001, Washington, DC)