Recursively saturated models of set theory and their close relatives

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Basic definitions and results (1)

Definition. Let $\mathcal{M} = (M, ...)$ be an $\mathcal{L}$-structure.
(a) $\mathcal{M}$ is *recursively saturated*, if for every $\mathcal{L}$-recursive type $p(x, \overline{y})$, if $p(x, \overline{m})$ is finitely realized for some $\overline{m}$ in M, then $p(x, \overline{m})$ is realized in $\mathcal{M}$.
(b) $\mathcal{M}$ is *resplendent*, if any Σ^1_1 sentence σ (with parameters in M) that holds in some elementary extension of $\mathcal{M}$, already holds in $\mathcal{M}$.

Remark. If $\mathcal{M}$ is a recursively saturated model of ZF, then:
(a) $\mathcal{M}$ is not ω-standard, since $p(x) = \{\text{“}x \in \omega\text{”}\} \cup \{x \text{ is not the } n\text{-th successor of } 0\text{: } n \in \omega\}$ is recursive and finitely realizable in $\mathcal{M}$.
(b) $\mathcal{M}$ has many undefinable elements, since the type $q(x)$ consisting of formule of the form $(\exists! v\ \varphi(v)) \to \neg\varphi(x)$, where $\varphi(v)$ ranges over all unary set-theoretic formulae, is recursive and finitely realizable in $\mathcal{M}$.

Theorem (Barwise and Schlipf).
(a) *Every resplendent structure is recursively saturated.*
(b) *Countable recursively saturated models are resplendent*.

Basic definitions and results (2)

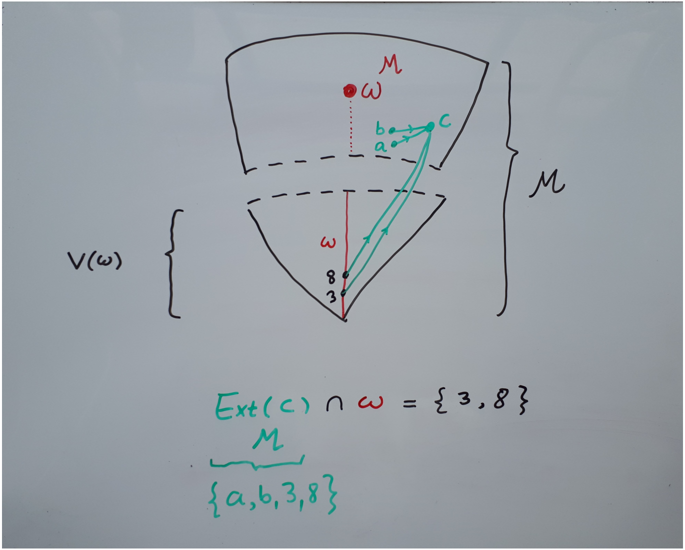
Definition. For $\mathcal{M} \models \mathrm{ZF}$ and $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, let $\mathcal{M}(\alpha) := (\mathrm{V}(\alpha), \in)^{\mathcal{M}}$.

Theorem. (Schlipf) *Let* $\mathcal{M} \models \mathrm{ZF}$.
$\mathcal{M}$ *is recursively saturated iff* $\mathcal{M}$ *is not* ω*-standard and the collection of* $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ *such that* $\mathcal{M}(\alpha) \prec \mathcal{M}$ *is unbounded in* $\mathrm{Ord}^{\mathcal{M}}$.
Proof. Exercise!

Theorem. *Recursive satuation is inherited by set-forcing extensions* (*E2002*), *and inner models (easy).*

Theorem (E2005) *Every countable model of* ZFC *has a class-generic extension that is pointwise definable. Therefore recursive saturation is not preserved by class forcing.*

A warm-up picture



Basic definitions and results (3)

Definition. Let $\mathcal{M}$ be a model of ZF and $c \in M$.
(a) $\mathrm{Ext}_{\mathcal{M}}(c) = \{m \in M : \mathcal{M} \models m \in c\}$. So if $\mathcal{M} = (M, \in)$ is a transitve model, $\mathrm{Ext}_{\mathcal{M}}(c) = c$.
(b) The *standard system* of a model $\mathcal{M}$ of ZF,denoted $\mathrm{SSy}(\mathcal{M})$, is $\{\mathrm{Ext}_{\mathcal{M}}(c) \cap \omega : c \in M\}$.

Theorem (essentially Wilmers). *The isomorphism type of a countable recursively saturated model $\mathcal{M}$ of ZF is determinded by* $\mathrm{Th}(M)$ and $\mathrm{SSy}(\mathcal{M})$.

Proof. Back-and-forth, based on two facts: (1) The type of each element of M (over finitely many parameters) is in $\mathrm{SSy}(\mathcal{M})$, (2) The types coded in $\mathrm{SSy}(\mathcal{M})$ that are finitely realized in $\mathcal{M}$ are realized in $\mathcal{M}$.

Putting Wilmers' theorem together with Schlipf's Theorem we obtain:

Theorem. *Suppose $\mathcal{M}$ is a countable recursively saturfated model of* ZF, *then the collection of $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that $\mathcal{M} \cong \mathcal{M}(\alpha) \prec \mathcal{M}$ is unbounded in* $\mathrm{Ord}^{\mathcal{M}}$.

A question of Paul Kindvall Gorbow

Gorbow's 2018 dissertation (University of Gothenburg) explores "Self-similarity in the Foundations", both in the orthodox context of ZF-style set theories and the unorthodox one of NF-style set theories. His results appear in JSL (2019) and AML(2019).

Last summer he asked me:

Question (Gorbow) *Is there an ω-standard model of ZFC such that the collection of $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that $\mathcal{M} \cong \mathcal{M}(\alpha) \prec \mathcal{M}$ is unbounded in $\mathrm{Ord}^{\mathcal{M}}$?*

We will see today that Gorbow's question has a positive answer. In the process, we will encounter the notions "condensable" and "cofinally condensable", which turn out to be close relatives of recursive saturation, these and other family relations will be explored in part (2) of the talk.

Infinitary Logic

Defintion.
(a) Let $\mathcal{L}_{\text{set}}$ be the usual vocabulary $\{=, \in\}$ of set theory. $\mathbb{L}_{\infty,\omega}$ is the infinitary language using the vocabulary $\mathcal{L}_{\text{set}}$ that allows conjunctions and disjunctions of *sets* (but not proper classes) of formulae, subject to the restriction that such infinitary formulae have at most *finitely many free variables*. Given a set Ψ of formulae, we denote such conjunctions and disjunctions respectively as $\bigwedge \Psi$ and $\bigvee \Psi$.
(b) $\mathbb{L}_{\delta,\omega}$ is the sublanguage of $\mathbb{L}_{\infty,\omega}$ that allows conjunctions and disjunctions of sets of formulae of cardinality *less than* δ. Note that $\mathbb{L}_{\omega,\omega}$ is none other than the usual first order language of set theory.
(c) Given $\mathbb{L} \subseteq \mathbb{L}_{\infty,\omega}$, and $\mathcal{L}_{\text{set}}$-structures $\mathcal{N}_1$ and $\mathcal{N}_2$, we write $\mathcal{N}_1 \prec_{\mathbb{L}} \mathcal{N}_2$ to indicate that $\mathcal{N}_1$ is a submodel of $\mathcal{N}_2$ and for every $\varphi(x_1, \cdots, x_n) \in \mathbb{L}$, and any n-tuple $(a_1, \cdots, a_n)$ from N_1, we have:

$$\mathcal{N}_1 \models \varphi(a_1, \cdots, a_n) \text{ iff } \mathcal{N}_2 \models \varphi(a_1, \cdots, a_n).$$

The well-founded part of a model of set theory

Definition. Suppose $\mathcal{M}$ is a model of ZF.

(a) $\mathrm{WF}(\mathcal{M})$ (the *well-founded part* of $\mathcal{M}$) consists of all elements m of $\mathcal{M}$ such that there is no infinite sequence $\langle a_n : n < \omega \rangle$ with $m = a_0$ and $a_{n+1} \in^{\mathcal{M}} a_n$ for all $n \in \omega$.

(b) Given $m \in M$, we say that *m is a nonstandard element of $\mathcal{M}$* if $m \notin \mathrm{WF}(\mathrm{M})$. It is well-known that if $\mathcal{M}$ is a model of ZF, then $\mathrm{WF}(\mathcal{M}) \subseteq_{\mathrm{rank}} \mathcal{M}$, and $\mathrm{WF}(\mathcal{M})$ satisfies KP (Kripke-Platek set theory).

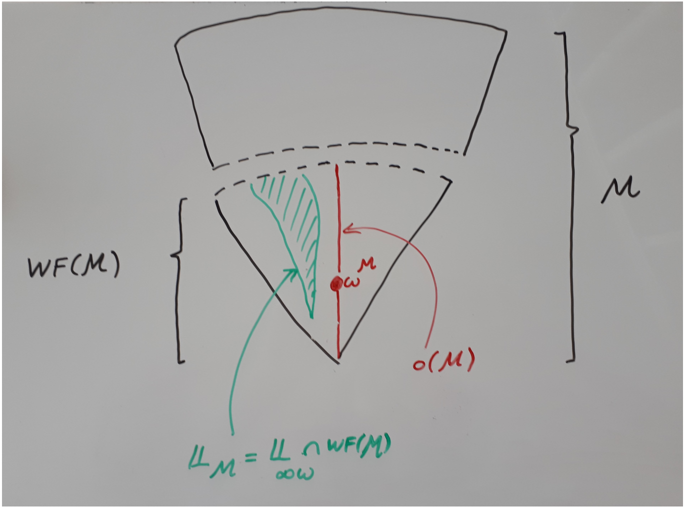
(c) We will identify $\mathrm{WF}(\mathcal{M})$ with its transitive collapse.

(d) $\mathbb{L}_{\mathcal{M}} = \mathbb{L}_{\infty,\omega} \cap \mathrm{WF}(\mathcal{M})$. Note that if M is countable,

$$\mathbb{L}_{\mathcal{M}} = \mathbb{L}_{\omega_1,\omega} \cap \mathrm{WF}(\mathcal{M}).$$

(e) $\mathrm{o}(\mathcal{M})$ (read as: *the ordinal of $\mathcal{M}$*) is the supremum of all ordinals that appear in the well-founded part of $\mathcal{M}$.

WF and friends in a picture



The main result of today's talk

Definition. (Condensable, cofinally condensable). A model $\mathcal{M}$ of ZF is *condensable,* if there is some $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that $\mathcal{M} \cong \mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$. $\mathcal{M}$ is *cofinally condensable* , there is an unbounded set of such αs in $\mathrm{Ord}^{\mathcal{M}}$.

Note that for an ω-nonstandard model $\mathcal{M}$ of ZF, $\mathbb{L}_{\mathcal{M}}$ is just the collection of (finitary) first order formulae, so the condition $\mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$ is equivalent to $\mathcal{M}(\alpha) \prec \mathcal{M}$ for ω-nonstandard models $\mathcal{M}$.

Theorem A. *Assuming a modest set-theoretic hypothesis, there is a countable model $\mathcal{M}$ of* ZFC *that is both* **definably well-founded** (*in the sense that every element of $\mathcal{M}$ that is first order definable in $\mathcal{M}$ is in the well-founded part of $\mathcal{M}$*), *and* **cofinally condensable**.

More definitions

Definition. Suppose $\mathcal{M}$ is a model of ZF.
(a) Given $\mathbb{L} \subseteq \mathbb{L}_{\infty,\omega}$, $\mathrm{Th}_{\mathbb{L}}(\mathcal{M})$ is the set of sentences (closed formulae) of $\mathbb{L}$ that hold in $\mathcal{M}$.
(b) For $\varphi \in \mathbb{L}_{\infty,\omega}$, the *depth* of φ, denoted $\mathrm{Depth}(\varphi)$, is the ordinal defined recursively by the following clauses:

(1) $\mathrm{Depth}(\varphi) = 0$, if φ is an atomic formula.
(2) $\mathrm{Depth}(\varphi) = \mathrm{Depth}(\psi) + 1$, if $\varphi = \neg\psi$.
(3) $\mathrm{Depth}(\varphi) = \mathrm{Depth}(\psi) + 1$, if $\varphi = \exists x\ \psi$.
(4) $\mathrm{Depth}(\varphi) = \sup\{\mathrm{Depth}(\psi) + 1 : \psi \in \Psi\}$, if $\varphi = \bigwedge \Psi$.
Within KP, one can code each formula φ with a set $\ulcorner\varphi\urcorner$, but in the interest of better readability we will often identify a formula with its code. This coding allows us to construe statements such as $\varphi \in \mathbb{L}_{\infty,\omega}$ and $\mathrm{Depth}(\varphi) = \alpha$ as statements in the first order language of set theory. The collection $\mathrm{D}(\alpha)$ of (codes of) $\mathbb{L}_{\infty,\omega}$-formule whose depth is less than α forms a set in ZF for all ordinals α, and $\mathbb{L}_{\mathcal{M}} = \bigcup_{\alpha \in \mathrm{o}(\mathcal{M})} \mathrm{D}^{\mathcal{M}}(\alpha)$.

Satisfaction classes (1)

Definition. (Satisfaction classes). Suppose $\mathcal{M}$ is a model of ZF, and $S \subseteq M$.

(a) For $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, S is an *α-satisfaction class* (over $\mathcal{M}$) if S correctly decides the truth of atomic sentences, and "S satisfies Tarski's compositional clauses of a truth predicate for $\mathrm{D}^{\mathcal{M}}(\alpha)$-sentences".
(b) S is an *∞-satisfaction class* over $\mathcal{M}$, if for every $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, S is an α-satisfaction class over $\mathcal{M}$.
(c) S is *separative* (over $\mathcal{M}$), if $(\mathcal{M}, S)$ satisfies the separation scheme $\mathrm{Sep(S)}$ in the extended language that includes a fresh predicate S (interpreted by S). We

now elaborate the meaning of (a) above. Reasoning within ZF, for each a in the universe of sets, let c_a be a constant symbol denoting a (where the map $a \mapsto c_a$ is Δ_1), and let $\mathrm{Sent}^+(\alpha, x)$ be the set-theoretic formula (with an ordinal parameter α and the free variable x) that defines the proper class of sentences of the form $\varphi(c_{a_1}, \cdots, c_{a_n})$, where $\varphi(x_1, \cdots, x_n) \in \mathrm{D}(\alpha)$ (the superscript $+$ on $\mathrm{Sent}^+(\alpha, x)$ indicates that x is a sentence in the language augmented with the indicated proper class of constant symbols). Then S is an α-satisfaction class over $\mathcal{M}$ if $(\mathcal{M}, S) \models \mathrm{Tarski}(\alpha, \mathrm{S})$, where $\mathrm{Tarski}(\alpha, \mathrm{S})$ is the (universal generalization of) the conjunction of the following axioms (I) through (IV).

Satisfaction classes (2)

(I) $\forall a \forall b \left(\left(\mathrm{S}\left(\ulcorner c_a = c_b \urcorner \right) \leftrightarrow a = b \right) \wedge \left(\mathrm{S}\left(\ulcorner c_a \in c_b \urcorner \right) \leftrightarrow a \in b \right) \right).$

(II) $\left(\mathrm{Sent}^{+}(\alpha, \varphi) \wedge (\varphi = \neg \psi) \right) \rightarrow (\mathrm{S}(\varphi) \leftrightarrow \neg \mathrm{S}(\psi)).$

(III) $\left(\mathrm{Sent}^{+}(\alpha, \varphi) \wedge (\varphi = \bigwedge \Psi) \right) \rightarrow (\mathrm{S}(\varphi) \leftrightarrow \forall \psi \in \Psi \ \mathrm{S}(\psi)).$

(IV) $\left(\mathrm{Sent}^{+}(\alpha, \varphi) \wedge \varphi = \exists x \ \psi(x) \right) \rightarrow (\mathrm{S}(\varphi) \leftrightarrow \exists x \ \mathrm{S}(\psi(c_x))).$

- In the interest of a lighter notation, if S is an α-*satisfaction class* over $\mathcal{M}$ and $\varphi(x_1, \cdots, x_n)$ is an n-ary formula of $\mathrm{D}^{\mathcal{M}}(\alpha)$ and $a_1, \cdots, a_n$ are in M, we will often write $\varphi\left(a_1, \cdots, a_n\right) \in S$ instead of $\varphi\left(c_{a_1}, \cdots, c_{a_n}\right) \in S$.

The following proposition is immediately derivable from the definitions involved.

Proposition. *If S is an α-satisfaction class over $\mathcal{M}$ for some nonstandard ordinal α of $\mathcal{M}$, then for all n-ary formula $\varphi(x_1, \cdots, x_n)$ of $\mathbb{L}_{\mathcal{M}}$ and all n-tuples $(a_1, \cdots, a_n)$ from M*, we have:

$$\mathcal{M} \models \varphi(a_1, \cdots, a_n) \text{ iff } \varphi(a_1, \cdots, a_n) \in S.$$

Satisfaction classes (3)

Remark. Reasoning within ZFC, given a limit ordinal γ, $(V(\gamma), \in)$ carries a separative γ-satisfaction class S since we can take S to be the Tarskian satisfaction class on $(V((\gamma), \in)$ for formulae of depth at most γ.

More specifically, the Tarski recursive construction/definition of truth works equally well in this more general context of infinitary languages since $(V(\gamma), \in)$ forms a set.

Observe that $(V(\gamma), \in, S) \models \mathrm{Sep}(\mathrm{S})$ comes "for free" since for any $X \subseteq V(\gamma)$ the expansion $(V(\gamma), \in, X)$ satisfies the scheme of separation in the extended language.

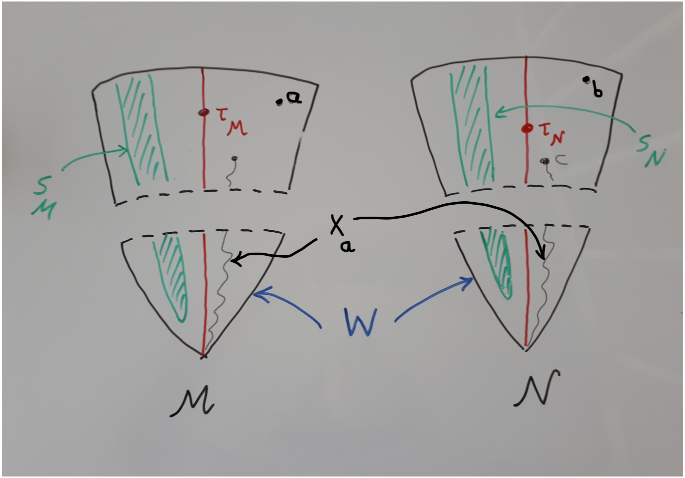
Overspill

The following proposition will be called upon in the proof of Theorem A.

Proposition. (Overspill) *Suppose $\mathcal{M}$ is a nonstandard model of ZF, and $S \subseteq M$ such that S is separative over $\mathcal{M}$. Assume furthermore that there is a first order formula $\theta(x, \overline{y})$ in the language $\{\in, \mathrm{S}\}$ and some sequence of parameters $\overline{m} \in M$ such that $(\mathcal{M}, S) \models \theta(\alpha, \overline{m})$ for every $\alpha \in \mathrm{o}(\mathcal{M})$. Then there is a nonstandard $\gamma \in \mathrm{Ord}^{\mathcal{M}}$ such that $(\mathcal{M}, S) \models \theta(\gamma, \overline{m})$.*

Proof. Suppose not, and let $A := \mathrm{WF}(\mathcal{M}) \cap \mathrm{Ord}^{\mathcal{M}}$. Then $A = \{x \in M : (\mathcal{N}, S) \models \theta(x, \overline{m}) \wedge \mathrm{Ord}(x)\}$. Since A is a bounded subset of $\mathrm{Ord}^{\mathcal{M}}$, by $\mathrm{Sep}(\mathrm{S})$, A is coded in $\mathcal{M}$, and therefore has a supremum σ in $\mathcal{M}$. This is a contradiction since $(\sigma, \in)^{\mathcal{M}}$ is well-founded, and yet $\sigma \notin A$ since A has no last element. $\square$

Another warm-up picture



The Isomorphism Lemma (1)

Isomorphism Lemma. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable nonstandard models of* ZFC *with the same well-founded part W, and let $\mathbb{L} := \mathbb{L}_{\mathcal{M}} = \mathbb{L}_{\mathcal{N}}$. Then $\mathcal{M} \cong \mathcal{N}$ if the following three conditions are satisfied*:

(*I*) $\mathrm{Cod}_W(\mathcal{M}) = \mathrm{Cod}_W(\mathcal{N})$. Here $\mathrm{Cod}_W(\mathcal{K}) = \{\mathrm{Ext}_{\mathcal{K}}(c) \cap W : c \in K\}$.

(*II*) $\mathrm{Th}_{\mathbb{L}}(\mathcal{M}) = \mathrm{Th}_{\mathbb{L}}(\mathcal{N})$.

(*III*) *For some* **nonstandard** *ordinals τ_M of $\mathcal{M}$, and τ_N of $\mathcal{N}$, there are $S_M \subseteq M$ and $S_N \subseteq N$ such that S_M is a separative τ_M-satisfaction class over $\mathcal{M}$, and S_N is a separative τ_N-satisfaction class over $\mathcal{N}$.*

Proof. The isomorphism between $\mathcal{M}$ and $\mathcal{N}$ can be built by a routine back-and-forth construction once we establish the claim below, for which we introduce the following convention:

- Given an n-tuple $\overline{a} = (a_0, \cdots, a_{n-1})$ from M (where $n \in \omega$), and an n-tuple $\overline{b} = (b_0, \cdots, b_{n-1})$ from N, we write $\overline{a} \sim \overline{b}$ as a shorthand for the following statement:

  for each n-ary formulae $\varphi(\overline{x})$ of $\mathbb{L}$, $\varphi(\overline{a}) \in S_M$ iff $\varphi(\overline{b}) \in S_N$.

The Isomorphism Lemma (2)

Claim. *Suppose* $\overline{a} \sim \overline{b}$. *Then*:

(i) *For every* $a \in M$ *there is some* $b \in N$ *such that* $(\overline{a}, a) \sim \left(\overline{b}, b\right)$.

(ii) *For every* $b \in N$ *there is some* $a \in M$ *such that* $(\overline{a}, a) \sim \left(\overline{b}, b\right)$.

By symmetry it suffices to verify part (i) of the Claim. Observe that since $\mathcal{M}$ and $\mathcal{N}$ share the same well-founded part W, we can fix an ordinal η such that $\eta = \mathrm{o}(\mathcal{M}) = \mathrm{o}(\mathcal{N})$, and $N(\alpha) = M(\alpha)$ for all $\alpha < \eta$. Given $a \in M$, let

$$X_a := \{\varphi(\overline{v}, v) : \varphi(\overline{v}, v) \text{ is an } (n+1)\text{-ary formula of } \mathbb{L}, \text{ and } \varphi(\overline{a}, a) \in S_M\}.$$

A routine argument shows that $X_a \in \mathrm{Cod}_W(\mathcal{M})$ (using the assumption that S_M is a separative τ_M-satisfaction class over $\mathcal{M}$ and τ_M is a nonstandard ordinal of $\mathcal{M}$). So by assumption (I), $X_a \in \mathrm{Cod}_W(\mathcal{N})$. Hence there is some $c \in M$ such that $X = W \cap \mathrm{Ext}_{\mathcal{N}}(c)$. For any $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, consider the elements c_α and d_α of N, such that the following holds in $\mathcal{N}$:

$$c_\alpha = \{x \in c : x \in \mathrm{V}(\alpha)\} \text{ and } d_\alpha = \{x \in \mathrm{V}(\alpha) : x \notin c\}.$$

Then for each $\alpha < \eta$, both c_α and $d_\alpha \in W$. Note that for each $w \in W$,

$$w = \mathrm{Ext}_{\mathcal{M}}(w) = \mathrm{Ext}_{\mathcal{N}}(w).$$

The Isomorphism Lemma (3)

The choice of c_α and d_α together with the compositional properties of S_M allows us to conclude:

(1) For all $\alpha \in \eta$

$$\overbrace{\left(\exists x \left(\left(\bigwedge_{\varphi(\overline{v},v)\in c_\alpha} \varphi(\overline{a},x)\right) \wedge \left(\bigwedge_{\varphi(\overline{v},v)\in d_\alpha} \neg\varphi(\overline{a},x)\right)\right)\right)}^{\psi_\alpha(\overline{a})} \in S_M.$$

Observe that $\psi_\alpha(\overline{x})$ is a formula of $\mathbb{L}$.

Putting (1) together with the assumption $\overline{a} \sim \overline{b}$ yields $\psi_\alpha(\overline{b}) \in S_N$, i.e.,

(2) For all $\alpha \in \eta$

$$\overbrace{\left(\exists x \left(\left(\bigwedge_{\varphi(\overline{v},v)\in c_\alpha} \varphi(\overline{b},x)\right) \wedge \left(\bigwedge_{\varphi(\overline{v},v)\in d_\alpha} \neg\varphi(\overline{b},x)\right)\right)\right)}^{\psi_\alpha(\overline{b})} \in S_N.$$

The Isomorphism Lemma (4)

The key observation at this point is that there is a first order formula $\theta(\mathrm{S}, x, y, \overline{z})$ in the language of set theory augmented with the predicate S such that (2) can be re-expressed as:

(3) For all $\alpha \in \eta,\ (\mathcal{N}, S_N) \models \theta(\mathrm{S}, \alpha, c, \overline{b})$.

By invoking Overspill in the expanded structure $(\mathcal{N}, S_N)$, there is some *nonstandard* ordinal γ of $\mathcal{N}$ such $(\mathcal{M}, S_N) \models \theta(\mathrm{S}, \gamma, c, \overline{b})$, i.e.,

$$(4)\ (\mathcal{N}, S_N) \models \mathrm{S}\left(\exists x \left(\left(\bigwedge_{\varphi(\overline{v},v)\in c_\gamma} \varphi(\overline{b}, x)\right) \wedge \left(\bigwedge_{\varphi(\overline{v},v)\in d_\gamma} \neg\varphi(\overline{b}, x)\right)\right)\right).$$

By coupling (4) together with the assumption that $(\mathcal{M}, S_M)$ satisfies the existential conjunct of $\mathrm{Tarski}(\delta, S)$, so the existential statement deemed true in (4) by the interpretation S_N of S is witnessed by some $a \in M$. This is the desired element $b \in N$, i.e., $(\overline{a}, a) \sim \left(\overline{b}, b\right)$. This concludes the proof of the claim, and therefore of the Lemma. $\square$

An Easy Lemma + Hutchinson's Theorem

Easy Lemma. (ZFC) *Let λ be a strongly inaccessible cardinal, $S \subseteq V_\lambda$, and let*

$$C := \{\delta < \lambda : (\mathrm{V}(\delta), \in, S \cap \mathrm{V}(\delta)) \prec (\mathrm{V}(\lambda), \in, S)\}.$$

Then C is closed and unbounded in κ.

Proof. A Skolem hull argument, very similar to the proof of the Reflection Theorem. $\square$

The following theorem was established by Hutchinson (1976) using the omitting types theorem. It can also be proved using generic ultrapowers for models of ZFC.

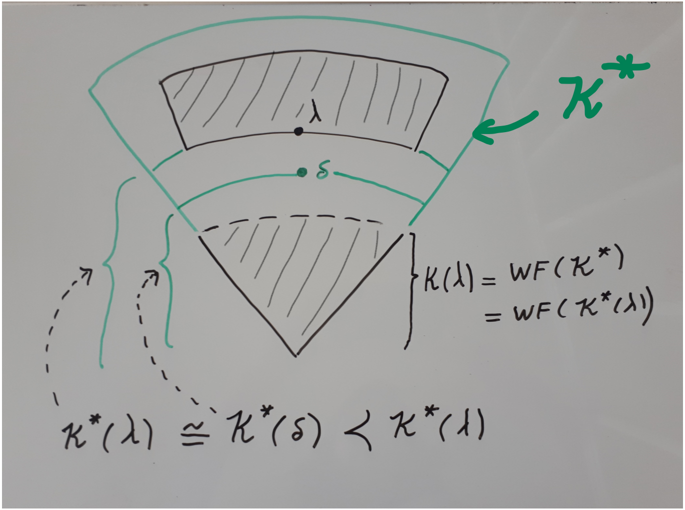
Theorem. (Hutchinson) *Suppose λ is a regular uncountable cardinal in a countable model $\mathcal{K}$ of* ZF. *Then there is an elementary extension $\mathcal{K}^*$ of $\mathcal{K}$ such that*:

(a) *$\mathcal{K}^*$ does not "perturb" any ordinal of $\mathcal{K}$ that is below λ, i.e., if $\mathcal{K} \models \alpha \in \lambda$, then $\mathrm{Ext}_{\mathcal{K}}(\alpha) = \mathrm{Ext}_{\mathcal{K}^*}(\alpha)$.*

(b) $\mathrm{Ext}_{\mathcal{K}^*}(\lambda) \setminus \mathrm{Ext}_{\mathcal{K}}(\lambda)$, *when ordered* by $\in^{\mathcal{K}^*}$, *has no first element.*

Remark. Condition (a) of Hutchinson's Theorem ensures that if $c \in K$ and $\mathcal{K} \models |c| < \lambda$, then $\mathcal{K}$ does not perturb c. Therefore, if $\mathcal{K}$ is well-founded and λ is strongly inaccessible in $\mathcal{K}$, then $\mathrm{WF}(\mathcal{K}^*)$ is precisely $K(\lambda)$.

Picture of Theorem A



The proof of Theorem A (1)

We are now ready to present the proof of Theorem A.

Theorem A. *Assuming that* ZFC + *"there exists an inaccessible cardinal" has a well-founded model, there is a model* $\mathcal{M}$ *of* ZFC *that is both definably well-founded and cofinally condensable.*

Proof. The proof is carried out in two steps, the first takes place within an appropriately chosen model $\mathcal{K}$ of ZFC; the second step is performed outside of $\mathcal{K}$.

Step 1. If the theory ZFC + "there exists an inaccessible cardinal" has a well-founded model, then by the Löwenheim-Skolem theorem and the fact that ZF proves that GCH holds in the constructible universe, there is a countable well-founded model that contains a strongly inaccessible cardinal. Let $\mathcal{K}$ be a countable well-founded model that contains a "cardinal" κ that is strongly inaccessible in the sense of $\mathcal{K}$. By collapsing $\mathcal{K}$ we may assume that $\mathcal{K} = (K, \in)$. We can use a Tarskian truth construction, together with the "Easy Lemma" to get hold of elements s and c of K satisfying the following conditions:

(i) $\mathcal{K} \models$ "s is a separative ∞-satisfaction class for $(\mathrm{V}(\lambda), \in)$".

(ii) $\mathcal{K} \models$ "c is unbounded in λ and $\forall \delta \in c\ (\mathrm{V}(\delta), \in, s \cap \mathrm{V}(\delta)) \prec (\mathrm{V}(\lambda), \in, s)$".

Proof of Theorem A (2)

Step 2. By Hutchinson's Theorem, there is an elementary extension $\mathcal{K}^*$ of $\mathcal{K}$ such that $\mathrm{WF}(\mathcal{K}^*) = K(\lambda)$.

We claim that $\mathcal{K}^*(\lambda)$ is definably well-founded and cofinally condensable. $\mathcal{K}^*(\lambda)$ is definably well-founded since $\mathrm{WF}(\mathcal{K}^*) = \mathcal{K}(\lambda) \prec \mathcal{K}^*(\lambda)$. To show that $\mathcal{K}^*(\lambda)$ is cofinally condensable, we will show that if $\delta \in \mathrm{Ext}_{\mathcal{K}^*}(c) \backslash K$, where c is as in (ii), then $\mathcal{K}^*(\kappa) \cong \mathcal{K}^*(\delta) \prec_{\mathbb{L}} \mathcal{K}$ for $\mathbb{L} = \mathbb{L}_{\mathcal{K}} = \mathcal{L}_{\mathcal{K}^*}$. If $S := \mathrm{Ext}_{\mathcal{K}^*}(s)$, then the assumption of the Isomorphism Lemma are satisfied with:

$$\mathcal{M} := \mathcal{K}^*(\lambda), \mathcal{N} := \mathcal{K}^*(\delta);\ \tau_M := \lambda,\ \tau_N := \delta,\ S_M := S, \text{ and } S_N := S \cap K^*(\delta).$$

(i) and (ii) ensure that $\mathcal{K}^*(\delta) \prec_{\mathbb{L}} \mathcal{K}$. $\square$

End of Part (1)

Theorem B. *The following are equivalent for a countable model $\mathcal{M}$ of* ZF.

(a) $\mathcal{M}$ *is condensable*.

(b) $\mathcal{M}$ *is cofinally condensable*.

(c) $\mathcal{M}$ *is nonstandard, and* $\mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$ *for an unbounded collection of* $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(d) $\mathcal{M}$ *is nonstandard and W-saturated, and* $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$.

(e) *For some nonstandard ordinal* γ *of* $\mathcal{M}$ *and some* $S \subseteq M$, *S is an amenable* γ-satisfaction class over $\mathcal{M}$.

W-saturation

Definition. Let $\mathcal{M}$ be a model of ZF, and $W = \mathrm{WF}(\mathcal{M})$. $\mathcal{M}$ is $\mathrm{Cod}_W(\mathcal{M})$-*saturated*, if for every type $p(x, y_1, \cdots, y_k)$, and for every k-tuple $\overline{a}$ of parameters from $\mathcal{N}$, $p(x, \overline{a})$ is realized in $\mathcal{M}$, provided the following three conditions are satisfied:

(i) $p(x, \overline{y}) \subseteq \mathbb{L}_{\mathcal{M}} = \mathbb{L}_{\infty,\omega} \cap \mathrm{WF}(\mathcal{M})$.

(ii) $p(x, \overline{y}) \in \mathrm{Cod}_W(\mathcal{M})$.

(iii) $\forall w \in W\ \mathcal{N} \models \exists x \left(\bigwedge_{\varphi \in p(x,\overline{y}) \cap w} \varphi(x, \overline{a}) \right)$.

We will say $\mathcal{M}$ is *W-saturated* instead of $\mathcal{M}$ is $\mathrm{Cod}_W(\mathcal{M})$-saturated.

Remark. An ω-nonstandard model $\mathcal{M}$ is W-saturated iff it is recursively saturated (since $\mathcal{M}$ is recursively saturated iff $\mathcal{M}$ is $\mathrm{SSy}(\mathcal{M})$-saturated).

Lemma. *If $\gamma \in \mathrm{Ord}^{\mathcal{M}} \backslash W$, then $\mathcal{M}(\gamma)$ is W-saturated.*

$\mathrm{ZF}(\mathbb{L})$

Given $\mathbb{L} \subseteq \mathbb{L}_{\infty,\omega}$, $\mathrm{ZF}(\mathbb{L})$ is the natural extension of ZF in which the scheme Sep of separation and Coll of collection are extended to the schemes $\mathrm{Sep}(\mathbb{L})$ and $\mathrm{Coll}(\mathbb{L})$ to allow formulae in $\mathbb{L}$ to be used for "separating" and "collecting" (respectively).

Examples.

(1) If $\mathcal{M}$ is ω-nonstandard, then $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$.

(2) If λ is strongly inaccessible in a model $\mathcal{N}$ of ZF, and $\mathcal{M} = \mathcal{N}(\lambda)$, then $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$.

(3) If $\mathcal{M}$ is pointwise definable, and $\mathrm{o}(\mathcal{M}) > \omega$, then $\mathcal{M}$ is **not** a model of $\mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$.

Elementary Chains and Reflection

Elementary Chains Theorem. *Suppose* $\mathbb{L} \subseteq \mathbb{L}_{\infty,\omega}$ *where* $\mathbb{L}$ *is closed under subformulae;* $(I, \lhd)$ *is a linear order;* $\langle \mathcal{M}_i : i \in I \rangle$ *is an* $\mathbb{L}$*-elementary chain of structures (i.e. ,* $\mathcal{M}_i \prec_{\mathbb{L}} \mathcal{M}_j$ *whenever* $i \lhd j$*); and* $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$*. Then* $\mathcal{M}_i \prec_{\mathbb{L}} \mathcal{M}$ *for each* $i \in I$*.*

Reflection Theorem. *Suppose* $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$, $\Phi \subseteq \mathbb{L}_{\mathcal{M}}$ *such that* $\Phi \in \mathrm{WF}(\mathcal{M})$*. For each n-ary formula* $\varphi \in \mathbb{L}_{\mathcal{M}}$ *let*:

$$\mathrm{Ref}_{\varphi}(\alpha) :=$$

$$\left(\forall x_1 \in \mathrm{V}(\alpha) \cdots \forall x_n \in \mathrm{V}(\alpha) \;\left(\varphi\left(x_1, \cdots, x_n\right) \longleftrightarrow \varphi^{\mathrm{V}(\alpha)}\left(x_1, \cdots, x_n\right)\right)\right).$$

Then $\mathcal{M} \models \forall \delta \in \mathrm{Ord}\ \exists \alpha \in \mathrm{Ord}\ \left((\delta \in \alpha) \wedge \bigwedge_{\varphi \in \Phi} \mathrm{Ref}_{\varphi}(\alpha)\right).$

The Isomorphism Lemma, Distilled

Distilled Isomorphism Lemma. *Given countable nonstandard models $\mathcal{M}$ and $\mathcal{N}$ of* ZF, *$\mathcal{M} \cong \mathcal{N}$ if the following conditions hold*:

(a) *$\mathcal{M}$ and $\mathcal{N}$ have the same well-founded part W.*

(b) $\mathrm{Cod}_W(\mathcal{M}) = \mathrm{Cod}_W(\mathcal{M})$.

(c) $\mathrm{ZF}(\mathbb{L}) \subseteq \mathrm{Th}_{\mathbb{L}}(\mathcal{M}) = \mathrm{Th}_{\mathbb{L}}(\mathcal{N})$ *for* $\mathbb{L} := \mathbb{L}_{\mathcal{M}} = \mathbb{L}_{\mathcal{N}}$.

(d) *Both $\mathcal{M}$ and $\mathcal{N}$ are W-saturated.*

The formulae $\mathrm{Sat}_\alpha(x)$

Proposition. *Suppose $\mathcal{M}$ is a model of* KP, *and $\alpha \in \mathrm{o}(\mathcal{M})$. Then there is some α-satisfaction class over $\mathcal{M}$ that is definable in $\mathcal{M}$ by an $\mathbb{L}_{\mathcal{M}}$-formula $\mathrm{Sat}_\alpha(x)$.*

Proof. The desired formula $\mathrm{Sat}_\alpha(x)$ is defined by the following recursion. A routine induction on α shows that $\mathrm{Sat}_\alpha(x)$ has the desired properties.

- $\mathrm{Sat}_1(x) :=$
  $\exists y \exists z\, [((x = \ulcorner c_y = c_z \urcorner) \wedge (y = z)) \vee ((x = \ulcorner c_y \in c_z \urcorner) \wedge (y \in z))]\,.$
- For $\alpha > 0$, $\mathrm{Sat}_\alpha(x) := \; [(\mathrm{Depth}(x) = 0) \wedge \mathrm{Sat}_1(x)] \wedge$

$\bigvee\limits_{0<\beta<\alpha} \big(\mathrm{Depth}(x) = \beta \wedge [\mathrm{Neg}_\beta(x) \vee \mathrm{Exist}_\beta(x) \vee \mathrm{Conj}_\beta(x)]\big)$, where;

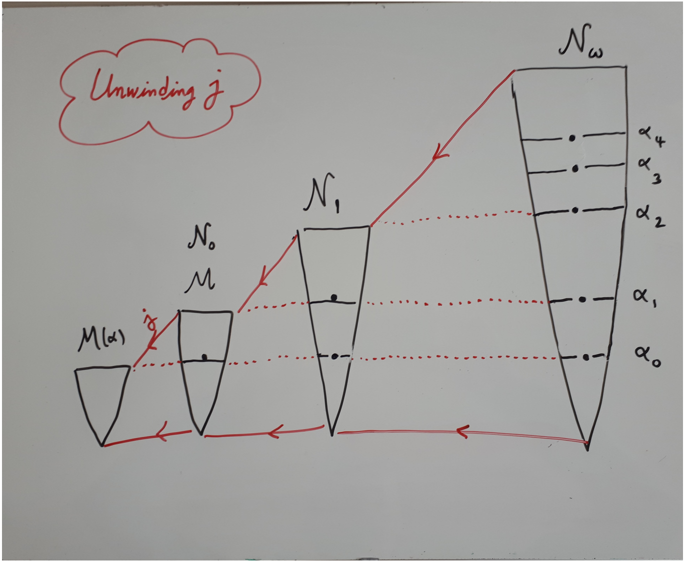
$$\mathrm{Neg}_\beta(x) := \exists y\, (x = \ulcorner \neg y \urcorner) \wedge \neg \mathrm{Sat}_\beta(y),$$

$$\mathrm{Exist}_\beta(x) := \exists y\; \exists v\, (x = \ulcorner \exists v\;\, y(v) \urcorner) \wedge \exists v\; \mathrm{Sat}_\beta(y(c_v)), \text{ and}$$

$$\mathrm{Conj}_\beta(x) := \exists y\, ((x = \ulcorner \textstyle\bigwedge y \urcorner)) \wedge (\forall z \in y \wedge \mathrm{Sat}_\beta(z)))\,.$$

□

Unwinding an elementary self-embedding



Condensable $\Rightarrow$ Cofinally Condensable (for ctble models)

Suppose $\mathcal{M}$ is condensable with $\mathcal{M} \cong \mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$. Note that $\mathcal{M}$ is W-saturated. Then by "unwinding" the isomorphism between $\mathcal{M}$ and $\mathcal{M}(\alpha)$, we can readily obtain a sequence of model $\langle \mathcal{N}_n : n \in \omega \rangle$ such that $\mathcal{N}_0 = \mathcal{M}$, and for all $n \in \omega$ the following hold:

(1) $\mathcal{N}_n \prec_{\mathrm{rank}} \mathcal{N}_{n+1}$ and $\mathcal{N}_n = (\mathrm{V}(\alpha_n), \in)^{\mathcal{N}_{n+1}}$ for some $\alpha_n \in \mathrm{Ord}(\mathcal{N}_{n+1})$.

(2) $\mathcal{N}_n \prec_{\mathbb{L}} \mathcal{N}_{n+1}$, where $\mathbb{L} := \mathbb{L}_{\mathcal{M}}$.

(3) $\mathcal{M} \cong \mathcal{N}_n$.

Let $\mathcal{N} := \bigcup\limits_{n \in \omega} \mathcal{N}_n$. By "Elementary Chains", $\mathcal{N}_n \prec_{\mathbb{L}} \mathcal{N}$ for all $n \in \omega$. Thus $\mathcal{N}$ is W-saturated, where $W = \mathrm{WF}(\mathcal{N}) = \mathrm{WF}(\mathcal{N}_n)$ for all $n \in \omega$. Thus $\mathcal{M}$ is also W-saturated. Therefore $\mathcal{N} \cong \mathcal{M}$ by the Distilled Isomorphism Lemma, which in light of (2) and (3) and the unboundedness of $\{\alpha_n : n \in \omega\}$ in $\mathrm{Ord}^{\mathcal{N}}$ makes it clear that (c) holds.

What happens to Theorem B if $\mathcal{M}$ is uncountable?

(1) *$\mathcal{M}$ is cofinally condensable*.

(2) *$\mathcal{M}$ is condensable*.

(3) *For some nonstandard ordinal γ of $\mathcal{M}$ and some $S \subseteq M$, S is an amenable* γ-satisfaction class over $\mathcal{M}$.

(4) *$\mathcal{M}$ is nonstandard, and $\mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$ for an unbounded collection of* $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(5) *$\mathcal{M}$ is nonstandard and W-saturated, and $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$*.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5).

We suspect that the implication (2) $\Rightarrow$ (1) fails for some uncountable model of ZF, but we have not been able to verify this. However, the remaining two implications can be shown to be irreversible.

The Barwise-Schlipf Theorem

- **Theorem** (Barwise-Schlipf) *The following are equivalent for a nonstandard model $\mathcal{M}$ of PA (of any cardinality).*

  (1) *$\mathcal{M}$ is recursively saturated.*

  (2) *There is $\mathfrak{X} \subseteq \mathcal{P}(M)$ such that $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{-}\mathrm{CA}_0$.*

  (3) $(\mathcal{M}, \mathrm{Def}(\mathcal{M}) \models \Delta^1_1\text{-}\mathrm{CA}_0 + \Sigma^1_1\text{-}\mathrm{AC}$.

- The Barwise-Schlipf proof of (1) $\Longrightarrow$ (3) uses Admissible Set Theory, and appears to be deep. A direct proof of this implication was found Feferman and Stavi (independently).

- The implication (3) $\Longrightarrow$ (2) is of course trivial. The proof of the implication (2) $\Longrightarrow$ (1) given by Barwise and Schlipf, is fairly short and plausible, but it has a nontrivial gap.

- The gap was only detected recently, and can be circumvented by machinery not available to Barwise and Schlipf (E-Schmerl, 2019).

- **Corollary.** $\Delta^1_1\text{-}\mathrm{CA}_0 + \Sigma^1_1\text{-}\mathrm{AC}$ *is a conservative extension of* PA.

Σ^1_1-AC

- Let $(Y)_x := \{y : \langle x, y \rangle \in Y\}$.
- Σ^1_1-AC is the scheme consisting of the formulae of the following form, where $\psi(x, X)$ is first order and is allowed to have parameters:

$$\forall x\ \exists X\ \psi(x, X) \to \exists Y\ \forall x\ \psi(x, (Y)_x).$$

- $\mathrm{GB} + \Sigma^1_1$-AC proves Global Choice.
- Σ^1_k-AC implies Δ^1_k-CA for all $k \in \omega$.

A Barwise-Schlipf Theorem for Set Theory

Theorem C. *The following are equivalent for a nonstandard model* $\mathcal{M}$ *of* ZF (*of any cardinality*).

(a) $\mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$ *for an unbounded collection of* $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(b) $(\mathcal{M}, \mathfrak{X}) \models \mathrm{GB} + \Delta^1_1\text{-}\mathrm{CA}$ *for* $\mathfrak{X} = \mathrm{Def}_{\mathbb{L}_{\mathcal{M}}}(\mathcal{M})$.

(c) *There is* $\mathfrak{X}$ *such that* $(\mathcal{M}, \mathfrak{X}) \models \mathrm{GB} + \Delta^1_1\text{-}\mathrm{CA}$.

Moreover, if $\mathcal{M}$ *is a* countable nonstandard model of ZFC *, then* (a) *and* (b) *are equivalent to*:

(d) *There is* $\mathfrak{X}$ *such that* $(\mathcal{M}, \mathfrak{X}) \models \mathrm{GB} + \Delta^1_1\text{-}\mathrm{CA} + \Sigma^1_1\text{-}\mathrm{AC}$.

Proof of Theorem C (1)

Proof. (**a**) $\Rightarrow$ (**b**). Assume (a). Then by $(c) \Leftrightarrow (d)$ of Theorem A, we have:

(1) $\mathcal{M}$ satisfies $\mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$, and (2) $\mathcal{M}$ is W-saturated.

(1) makes it clear that GB holds in $(\mathcal{M}, \mathfrak{X})$. We will use (2) to show that Δ^1_1-CA holds in $(\mathcal{M}, \mathfrak{X})$. To this end, let $U \subseteq M$ such that U is defined in $(\mathcal{M}, \mathfrak{X})$ by a Σ^1_1-formula $\exists X\ \psi^+(X, B, x)$, and $M \backslash U$ is defined in $(\mathcal{M}, \mathfrak{X})$ by a Σ^1_1-formula $\exists X\ \psi^-(X, B, x)$, where $B \in \mathfrak{X}$ is a class parameter definable by the $\mathbb{L}_{\mathcal{M}}$-formula $\beta(m, v)$ ($m \in M$ is a set parameter; note that we may assume without loss of generality that the only parameter in ψ^+ and in ψ^- is a class parameter B). Consider the infinitary formulae $\theta^+(x)$ and $\theta^-(x)$ defined as follows:

$$\theta^+(x) := \bigvee_{\varphi(y,v) \in \mathbb{L}_{\mathcal{M}}} \exists y\ \psi^+(X/\varphi(y, v), B/\beta(m, v), x), \text{ and}$$

$$\theta^-(x) := \bigvee_{\varphi(y,v) \in \mathbb{L}_{\mathcal{M}}} \exists y\ \psi^-(X/\varphi(y, v), B/\beta(m, v), x),$$

Proof of Theorem C (2)

By design:

(3) $\mathcal{M} \models \forall x\, (\theta^+(x) \vee \theta^-(x))$.

(4) **Claim.** There is some $\alpha \in \mathrm{o}(\mathcal{M})$ such that $\mathcal{M} \models \forall x\, (\theta^+_\alpha(x) \vee \theta^-_\alpha(x))$, where

$$\theta^+_\alpha(x) := \bigvee_{\varphi(y,v) \in \mathbb{L}_\mathcal{M} \cap M(\alpha)} \exists y\ \psi^+(X/\varphi(y), B/\beta(m,v), x),$$

$$\theta^-_\alpha(x) := \bigvee_{\varphi(y,v) \in \mathbb{L}_\mathcal{M} \cap M(\alpha)} \exists y\ \psi^-(X/\varphi(y), B/\beta(m,v), x).$$

Notice that (4) implies that U is definable in $\mathcal{M}$ by $\theta^+_\alpha(x)$, so the verification of Δ^1_1-CA will be complete once we establish (4), thanks to (1) and the fact that $\theta^+_\alpha(x) \in \mathbb{L}_\mathcal{M}$. To establish (4) we argue by contradiction. Suppose

(5) $\mathcal{M} \models \exists x\ \neg\, (\theta^+_\alpha(x) \vee \theta^-_\alpha(x))$ for each $\alpha \in \mathrm{o}(\mathcal{M})$.

Consider the $\mathbb{L}_\mathcal{M}$-type $p(x)$, where

$$p(x) := \{\neg\, (\theta^+_\alpha(x) \vee \theta^-_\alpha(x)) : \alpha \in \mathrm{o}(\mathcal{M})\}.$$

It is easy to see that $p(x) \in \mathrm{Cod}_W(\mathcal{M})$. By (5), for each $\alpha \in \mathrm{o}(\mathcal{M})$, $p(x) \cap M(\alpha)$ is realized in $\mathcal{M}$, so by W-saturation of $\mathcal{M}$, $p(x)$ is realized in $\mathcal{M}$, i.e., $\mathcal{M} \models \exists x\ \neg\, (\theta^+(x) \vee \theta^-(x))$, which contradicts (3) and finishes the proof of (4) $\square$

Proof of Theorem C (3)

(**c**) $\Rightarrow$ (**a**) follows from the lemma below.
Lemma $\diamondsuit$. *If* $(\mathcal{M}, \mathfrak{X}) \models \mathrm{GB} + \Delta^1_1\text{-}\mathrm{CA}$, *then the following hold*:

(a) $\mathrm{Sat}^{\mathcal{M}}_{\alpha} \in \mathfrak{X}$ *for each* $\alpha \in \mathrm{o}(\mathcal{M})$.

(b) $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$.

(c) *If* $\mathcal{M}$ *is nonstandard, then* $\mathcal{M}(\alpha) \prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M}$ *for an unbounded collection of* $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

Proof. (a) is proved by induction on α to verify that $\mathrm{Sat}^{\mathcal{M}}_{\alpha}$ is Δ^1_1-definable in $(\mathcal{M}, \mathfrak{X})$ for each $\alpha \in \mathrm{o}(\mathcal{M})$. For each $m \in M$ we have:

$$m \in \mathrm{Sat}^{\mathcal{M}}_{\alpha+1} \text{ iff}$$

$$(\mathcal{M}, \mathfrak{X}) \models \exists S\, [\mathrm{Sat}(S, \alpha) \wedge (\mathrm{Depth}(m) \leq \alpha) \wedge (\mathrm{Neg}(m) \vee \mathrm{Exist}(m) \vee \mathrm{Conj}(m))],$$

where

$$\mathrm{Neg}(x) := \exists y\, (x = \ulcorner \neg y \urcorner) \wedge \neg S(y);$$

$$\mathrm{Exist}(x) := \exists y\ \exists v\, (x = \ulcorner \exists v\ y(v) \urcorner) \wedge \exists v\ S(y(c_v)); \text{ and}$$

$$\mathrm{Conj}(x) := \exists y\, ((x = \ulcorner \textstyle\bigwedge y \urcorner)) \wedge (\forall z \in y\ S(z)))\,.$$

Proof of Theorem C (4)

Similarly, for each $m \in M$ we have:

$$m \in \mathrm{Sat}^{\mathcal{M}}_{\alpha+1} \text{ iff}$$
$$(\mathcal{M}, \mathfrak{X}) \models$$
$$\forall S \left((\mathrm{Sat}(S, \alpha) \wedge (\mathrm{Depth}(m) \leq \alpha)) \rightarrow (\mathrm{Neg}(m) \vee \mathrm{Exist}(m) \vee \mathrm{Conj}(m)) \right).$$

Thus $\mathrm{Sat}^{\mathcal{M}}_{\alpha+1}$ has both a Σ^1_1 and a Π^1_1 definition in $(\mathcal{M}, \mathfrak{X})$. The limit case is more straightforward since for limit α the following hold for each $m \in M$:

$$m \in \mathrm{Sat}^{\mathcal{M}}_{\alpha} \text{ iff}$$
$$(\mathcal{M}, \mathfrak{X}) \models \exists \beta < \alpha \ \left((\mathrm{Depth}(m) = \beta) \wedge \exists S \left(\mathrm{Sat}(S, \beta + 1) \wedge S(m) \right) \right), \text{ and}$$

$$m \in \mathrm{Sat}^{\mathcal{M}}_{\alpha} \text{ iff}$$
$$(\mathcal{M}, \mathfrak{X}) \models \exists \beta < \alpha \ \left((\mathrm{Depth}(m) = \beta) \wedge \forall S \left(\mathrm{Sat}(S, \beta + 1) \rightarrow S(m) \right) \right).$$

This concludes the proof of (a).
Note that (b) is an immediate consequence of (a), so we next proceed to demonstrate (c).

Proof of Theorem C (5)

Suppose (c) fails and let $\mathcal{M}$ be a nonstandard model of ZF such that $(\mathcal{M}, \mathfrak{X}) \models \mathrm{GB} + \Delta^1_1\text{-CA}$ and the collection of α such that $\mathcal{M}(\alpha) \prec \mathcal{M}$ is not cofinal in $\mathrm{Ord}^{\mathcal{M}}$. We will show that each of the following cases leads to a contradicion.

Case A. $\mathrm{o}(\mathcal{M}) = \omega$. This case can be handled by a strategy identical to the proof (given in the recent paper of Schmerl and myself) of the "problematic" direction of Barwise-Schlipf Theorem for models of PA.

Case B. $\mathrm{o}(\mathcal{M}) > \omega$. First note that by there is no $S \in \mathfrak{X}$ such that S is a γ-satisfaction class over $\mathcal{M}$ for any nonstandard $\gamma \in \mathrm{Ord}^{\mathcal{M}}$. Together with part (a) of Lemma $\diamondsuit$ we therefore have:

(1) $\forall \delta \in \mathrm{Ord}^{\mathcal{M}} \ (\delta \in \mathrm{o}(\mathcal{M}) \iff \exists S \in \mathfrak{X} \ \ (\mathcal{M}, S) \models \mathrm{Sat}(S, \delta))$.

Also, thanks to Theorem B, the failure of (c) assures us:

(2) $\exists \beta \in \mathrm{Ord}^{\mathcal{M}} \ \forall \delta \in \mathrm{Ord}^{\mathcal{M}}(\beta < \delta \Rightarrow \mathcal{M}(\delta) \not\prec_{\mathbb{L}_{\mathcal{M}}} \mathcal{M})$.

For each $\alpha \in \mathrm{o}(\mathcal{M})$ let $\mathbb{L}_\alpha := \mathbb{L}_{\infty,\omega} \cap \mathrm{V}(\alpha)$. Since $\mathcal{M} \models \mathrm{ZF}(\mathbb{L}_{\mathcal{M}})$, by Reflection Theorem, there is (in the real world) a sequence $\langle \gamma_\alpha : \alpha \in \mathrm{o}(\mathcal{M}) \rangle$ of ordinals of $\mathcal{M}$ such that for each $\alpha \in \mathrm{o}(\mathcal{M})$ the following holds:

$$\mathcal{M} \models \text{“}\gamma_\alpha \text{ is the first ordinal } \gamma > \beta \text{ such that } \mathrm{V}(\gamma) \prec_{\mathbb{L}_\alpha} \mathrm{V}\text{”}.$$

Let $\Gamma = \{\gamma_\alpha : \alpha \in \mathrm{o}(\mathcal{M})\}$. There are two cases to consider.

Case B1. $\mathrm{o}(\mathcal{M}) > \omega$ and Γ is cofinal in $\mathrm{Ord}^{\mathcal{M}}$. Let $F = \{\langle \alpha, \gamma_\alpha \rangle : \alpha \in \mathrm{o}(\mathcal{M})\}$. By (1), F is defined by the following Σ^1_1-formula $\varphi(\delta, \gamma)$

$$\varphi^+(\delta,\gamma) :=$$

$$(\delta \in \mathrm{Ord}) \wedge (\gamma \in \mathrm{Ord}) \left[\begin{array}{c} \exists X \in \mathfrak{X}\ \mathrm{Sat}(X, \alpha+\omega) \left(\bigwedge\limits_{\varphi \in \mathbb{L}_\delta} \mathrm{Ref}_\varphi(\gamma) \right) \in S \\ \wedge\ \forall \gamma' < \gamma \left(\bigwedge\limits_{\varphi \in \mathbb{L}_\delta} \mathrm{Ref}_\varphi(\gamma) \right) \notin S \end{array} \right].$$

Note that $\mathrm{Depth} \left(\bigwedge\limits_{\varphi \in \mathbb{L}_\delta} \mathrm{Ref}_\varphi(\gamma) \right) < \delta + \omega$. To see that the complement of F is also Σ^1_1-definable in $(\mathcal{M}, \mathfrak{X})$, we observe that $\langle \gamma_\alpha : \alpha \in \mathrm{o}(\mathcal{M}) \rangle$ is a strictly increasing sequence, and Γ is a closed subset of $\mathrm{Ord}^{\mathcal{M}}$, i.e., for limit $\beta \in \mathrm{o}(\mathcal{M})$, $\gamma_\beta = \sup\{\gamma_\alpha : \alpha < \beta\}$. Thus for each $\delta \in \mathrm{Ord}^{\mathcal{M}} \backslash \Gamma$, either there *delta* $< \gamma_0$, or there is some $\alpha \in \mathrm{o}(\mathcal{M})$ such that $\gamma_\alpha < \delta < \gamma_{\alpha+1}$.

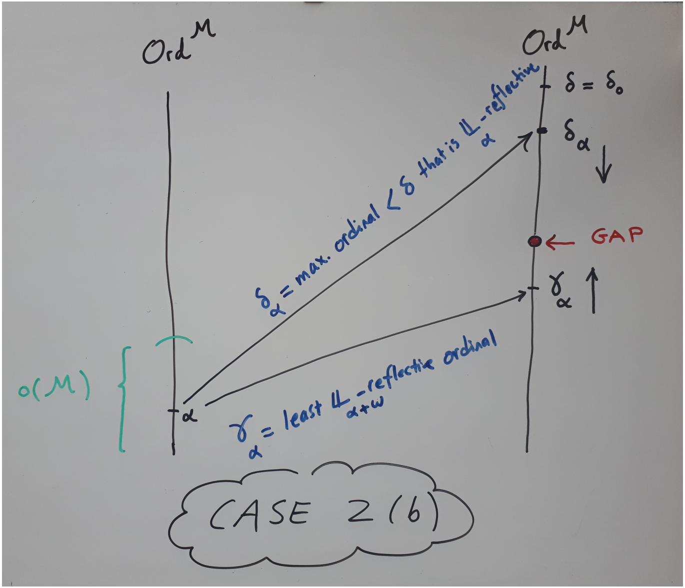
Proof of Theorem C (7)

So the following Σ^1_1-formula $\varphi^-(\delta,\gamma)$ defines the complement of F:

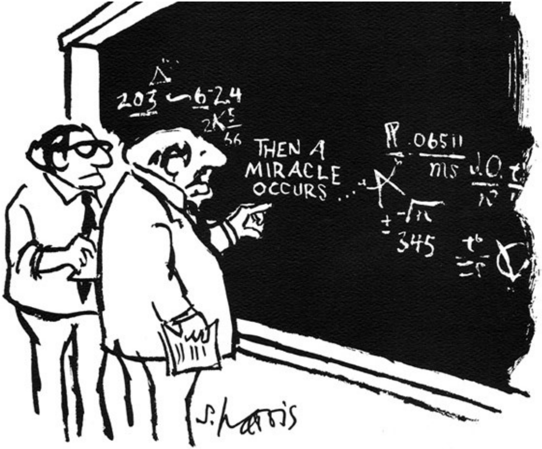
$$\varphi^-(\delta,\gamma) := ((\delta \in \mathrm{Ord}) \wedge (\gamma \in \mathrm{Ord})) \to [\delta < \gamma_0 \vee \exists \alpha \in \mathrm{Ord}\ \ (\gamma_\alpha < \delta < \gamma_{\alpha+1})].$$

Therefore $F \in \mathfrak{X}$, which contradicts the veracity of Collection in $(\mathcal{M}, F)$. This takes care of case B1.
Case B2. $\mathrm{o}(\mathcal{M}) > \omega$ and Γ is bounded in $\mathrm{Ord}^{\mathcal{M}}$. In this case, by (2) the supremum of Γ does not exists in $\mathrm{Ord}^{\mathcal{M}}$. Let $\delta \in \mathrm{Ord}^{\mathcal{M}}$ be an upper bound for Γ. In the real world define $\langle \delta_\alpha : \alpha \in \mathrm{o}(\mathcal{M})\rangle$ with $\delta_0 = \delta$ and $\delta_{n+1} =$ greatest ordinal below δ that is $\mathbb{L}_\alpha$-reflective. It is readily seen that:
(1) $\delta_\beta > \delta_\alpha$ if $\alpha < \beta \in \mathrm{o}(\mathcal{M})\backslash\mathrm{WF}(\mathcal{M})$, and
(2) $\{\delta_\alpha : \alpha \in \mathrm{o}(\mathcal{M})\}$ has no least element.
Moreover, there is a sequence $\langle \psi_\alpha(x) : \alpha \in \mathrm{o}(\mathcal{M})\rangle$ of $\mathbb{L}_{\mathcal{M}}$-formulae that is in $\mathrm{Cod}_W(\mathcal{M})$ such that $\psi_\alpha(x)$ defines δ_α in $\mathcal{M}$. We then use a trick similar to the ω-nonstandard case and to show that a proper cut I of $\mathrm{Ord}^{\mathcal{M}}$ is in $\mathfrak{X}$, thereby arriving at a contradiction, which concludes our verification of (c). $\square$

Case 2(b) Illustrated



Thank you for your attention



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."