The Barwise-Schlipf characterization of recursively saturated models of PA

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Our story begins with ...

- The following seminal paper inaugurated the study of recursively saturated models of PA.

  J. Barwise and J. Schlipf, *On recursively saturated models of arithmetic*, in: **Model theory and algebra** (*A memorial tribute to Abraham Robinson*), Lecture Notes in Math., vol. 498, 42–55, Springer, 1975.

- And the following papers did much to "spread the word":

  R. Murawski, *On expandability of models of Peano arithmetic*. **I**, **II**, **III**, Studia Logica, vol. 35, pp. 409-419 and 421-431; vol. 36, pp. 181-188; correction: Studia Logica, vol. 36 (1976/1977).
  C. Smoryński, *Recursively saturated nonstandard models of arithmetic*, **J. Symb. Logic**, (1981), 259-286.

First page of the Barwise-Schlipf paper

ON RECURSIVELY SATURATED MODELS OF ARITHMETIC[1]

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§1. Introduction. In his retiring presidential address to the ASL Abraham Robinson pointed out that one of the legitimate functions of the logician is "to use his own characteristic tools ... to gain a better understanding of the various and varigated kinds of structures, methods, theories and theorems that are to be found in mathematics" ([6], p. 500). In this note we use our characteristic tools, admissible sets with urelements from Barwise [1] and recursively saturated models from Schlipf [7], to shed a glimmer of light on the models that arise in non-standard analysis and some of the known theorems about them.

1.1 Definition. Let $\mathfrak{M} = \langle M, R_1, \ldots, R_k \rangle$ be a structure for a finite language L. We say that $\mathfrak{M}$ is recursively saturated if for every recursive set $\Phi(x, y_1, \ldots, y_n)$ of finitary formulas of L, the following infinite sentence is true in $\mathfrak{M}$:

$$\forall y_1 \ldots y_n [\bigwedge_{\Phi_0 \in S_\omega(\Phi)} \exists x \bigwedge \Phi_0(x, \vec{y}) \Rightarrow \exists x \bigwedge \Phi(x, \vec{y})]$$

where $S_\omega(\Phi)$ is the set of finite subsets of Φ.

It is not too hard to see that any model of Peano arithmetic (PA) which occurs as the integers in some model of non-standard analysis (or in some non ω-model of ZF) is recursively saturated. The principle goal of this paper is to:

(a) isolate a weak subsystem of analysis, called Δ^1_1-PA

(b) prove that the recursively saturated models of PA are exactly those models that can be expanded to models of Δ^1_1-PA

(c) derive certain corollaries from (b).

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4. Epilogue. We conclude by making some (perhaps controversial) remarks on subsystems of analysis.

Let Δ^1_1-CA be the theory Δ^1_1-PA plus the full second-order scheme of induction. Δ^1_1-CA is not a conservative extension of PA since, for example, Con(PA) is provable in Δ^1_1-CA but not in PA. The theory Δ^1_1-CA, and stronger theories like π^1_1-CA, have been studied extensively by proof theoretic methods, but there does not seem to be a good model theory of such subsystems. Our Theorem 1.2, on the other hand, shows that Δ^1_1-PA does have an interesting model theory. So it seems to suggest that the study of other subsystems of analysis, and their associated model theory, might proceed more fruitfully with the axiom of induction, rather than scheme.

Moral: make your induction match your comprehension.

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Birth of Recursive Saturation

Recursive saturation, as a general concept, made its debut in the following sources:

- J. Barwise, **Admissible sets and structures**, Springer-Verlag, 1975.
- J. Barwise and J. Schlipf, *An introduction to recursively saturated and resplendent models*, **J. Symb. Logic** 41 (1976), 531–536.
- J. Schlipf, *A guide to the identification of admissible sets above structures*, **Ann. Math. Logic** 12 (1977), 151–192.
- J. Schlipf, *Toward model theory through recursive saturation*, **J. Symb. Logic** 43 (1978), 183–206.
- J.-P. Ressayre, *Models with compactness properties relative to an admissible language*, **Ann. Math. Logic** 11 (1977), 31–55.

Excerpt of Barwise and Schlipf's account (1)

In early 1972, Barwise began reworking the theory of admissible sets so as to allow them to be built up out of mathematical structures, rather than just out of the empty set. One of the features that soon emerged was that many infinite structures $\mathcal{M}$ could now be elements of admissible sets A with $o(A) = \omega$ e.g., this holds if $\mathcal{M}$ is ω-saturated. It was also clear that such structures had very nice model theoretic properties, by means of the associated infinitary completeness and compactness theorems. In the summer of 1973 Schlipf introduced the notion of recursively saturated structure, and proved that they are precisely those with $o(\mathrm{HYP}(\mathcal{M})) = \omega$. This gave, retroactively, a great many interesting facts about countable, recursively saturated models, including pseudo-uniqueness and co-homogeneity.

Excerpt of Barwise and Schlipf's account (2)

In the winter of 1973, Ressayre circulated some handwritten notes on his notion of L_A-Σ-compact structure, again where A is an admissible set of height greater than ω. Harnik and Makkai, familiar with admissible sets with urelements and Schlipf's Theorem, translated Ressayre's notion into a simpler equivalent in terms of admissible sets with urelements. If you take their version of Ressayre's notion and restrict it to admissible sets of height ω, you get the notion of recursively saturated structure.

Excerpt of Ressayre's account

As for any natural notion, there are many paths leading to recursively saturated models. Infinitary model theory is one of them, which brings these models down from the sky. Suppose you want to extend to $\mathcal{L}_{\omega_1,\omega}$ the method of saturated models; clearly compactness is needed, but the Barwise compactness theorem applies only to Σ theories and yields only models that are saturated with respect to Σ types. Thus you have to content yourself with this weak saturation property called Σ-saturation (which would not be the case if you dealt with finitary logic only). This constraint makes it much easier to realize that through resplendence, Σ-saturation implies some of the main consequences of saturation; and in the particular case of $\mathcal{L}_{\omega,\omega}$, you thus get the (countable) recursively saturated models and their resplendence. These considerations did in fact lead to the first work on recursively saturated models, as if the infinitary detour were a short cut.

Excerpt of Smoryński's account (1)

Through the ability of arithmetic to partially define truth and the ability of infinite integers to simulate limit processes, nonstandard models of arithmetic automatically have a certain amount of saturation: Any encodable partial type whose formulae all fall into the domain of applicability of a truth definition must, by finite satisfiability and Overspill, be nonstandard-finitely satisfiable-whence realized. This fact was first exploited by A. Robinson[1963] who used the unrealizability in a given model of a certain encodable partial type to prove Tarski's Theorem on the Undefinability of Truth. A decade later, H. Friedman brought this phenomenon to the public's attention by using it to establish impressive embeddability criteria for countable nonstandard models of arithmetic. Subsequently, Wilkie considered models expandable to "strong theories" and, among such models, complemented Friedman's embeddability criteria with elementary embeddability and isomorphism criteria. Oddly enough, the fact that some kind of saturation property was being employed was not explicitly acknowledged in any of this work.

Excerpt of Smoryński's account (2)

The study of recursively saturated models of arithmetic has another starting point-namely, questions of the expandability of models of arithmetic to models of stronger theories. The prehistory of this approach begins again in the 1960s, when Ehrenfeucht and Kreisel gave an example of nonexpandability by means of an argument closely allied to that cited above of Robinson: A truth definition for arithmetic entails the existence of much larger elements than would necessarily exist in a model not having such a truth definition. The general introduction of recursive saturation into model theory brought with it a general positive expandability result-the strong relation universality, or resplendence, of countable recursively saturated models. (Cf. Ressayre [1977] or Schlipf [1977].) It also brought with it a specific expandability result (Barwise and Schlipf [1975]): A model of arithmetic is recursively saturated iff it is expandable to a weak second-order theory with an induction axiom and a comprehension (or even choice) schema.

The Barwise-Schlipf characterization

Theorem (Barwise-Schlipf) *The following are equivalent for a nonstandard model $\mathcal{M}$ of PA (of any cardinality).*

(1) $\mathcal{M}$ *is recursively saturated.*

(2) *There exists $\mathfrak{X} \subseteq \mathcal{P}(M)$ such that $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{-}\mathrm{CA}_0$.*

(3) $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \Delta^1_1\text{-}\mathrm{CA}_0 + \Sigma^1_1\text{-}\mathrm{AC}$.

The notation explained

- $\mathrm{Def}(\mathcal{M})$ is the collection of subsets of M that are parametrically definable in $\mathcal{M}$.
- ACA_0 is the theory formulated in the two-sorted language $\mathcal{L}_2$ of second order arithmetic (one sort for numbers, the other for sets of numbers) whose axioms consist of PA^-, the induction **axiom**:

$$\forall X([0 \in X \wedge \forall x(x \in X \rightarrow x+1 \in X)] \rightarrow \forall x(x \in X)),$$

  and the arithmetical comprehension scheme consisting of formulae of the following form where $\psi(x, X)$ is first order and is allowed to have parameters:

$$\exists X \forall x(x \in X \leftrightarrow \psi(x, X)).$$

Remarks about ACA_0

- $(\mathcal{M}, \mathfrak{X}) \models \mathrm{ACA}_0$ iff (1) and (2), where:
  (1) $(\mathcal{M}, X)_{X \in \mathfrak{X}} \models \mathrm{PA}^*$.
  (2) If $X \in \mathfrak{X}$, then $\mathrm{Def}(\mathcal{M}, X) \subseteq \mathfrak{X}$.
- Therefore $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \mathrm{ACA}_0$ for every model $\mathcal{M}$ of PA, which shows that ACA_0 is conservative over PA.
- However, in contrast to PA, ACA_0 is finitely axiomatizable.
- ACA_0 is not interpretable in PA, and has superexponential speed-up over PA.

- A Σ^1_1-formula is of the form $\exists X\ \varphi(X, x)$, and a Π^1_1-formula is a formula of the form $\forall X\ \varphi(X, x)$, where $\varphi(X, x)$ is arithmetical.
- Δ^1_1-CA_0 is the extension of ACA_0 in which the arithmetical comprehension is extended to Δ^1_1-CA, i.e., the scheme scheme whose instances are of the following form, where $\sigma(x)$ is a Σ^1_1-formula and $\pi(x)$ is a Π^1_1-formula (set parameters allowed in both $\sigma(x)$ and $\pi(x)$)

$$\forall x\, [\sigma(x) \leftrightarrow \pi(x)] \longrightarrow \exists X \forall x [x \in X \leftrightarrow \sigma(x)].$$

Σ^1_1-AC and Σ^1_1-Coll

- Let $(Y)_x := \{y : p(x,y) \in Y\}$, where $p(x,y)$ is a pairing function.
- Σ^1_1-AC is the scheme consisting of the formulae of the following form, where $\psi(x,X)$ is first order and is allowed to have parameters:

$$\forall x\ \exists X\ \psi(x,X) \to \exists Y\ \forall x\ \psi(x,(Y)_x).$$

- Σ^1_1-Coll is the scheme consisting of formulae of the following form, where $\psi(x,X)$ is first order and is allowed to have parameters:

$$\forall x\ \exists X\ \psi(x,X) \to \exists Y\ \forall x\ \exists y\ \psi(x,(Y)_y).$$

- It is easy to see that in the presence of ACA_0, Σ^1_1-Coll is equivalent to Σ^1_1-AC.
- Also, it known that Σ^1_k-AC implies Δ^1_k-CA for all $k \in \omega$; an easy proof can be found in Simpson's SOSOA. Apparently, when Barwise and Schlipf were writing their 1975 paper, they were unaware of this, but by the time Smoryński wrote his 1981 paper, this became well known, as he describes it as "evident" that Σ^1_1-AC_0 implies Δ^1_1-CA_0.

Corollaries of the Barwise-Schlipf Theorem

- **Corollary 1.** $\Delta^1_1\text{-CA}_0 + \Sigma^1_1\text{-AC}$ *is a conservative extension of* PA.
- **Corollary 2.** *Suppose* $\mathcal{M}$ *is a nonstandard model of* PA. *If* $\mathcal{M}$ *is rec. sat., then* $\mathcal{M}$ *has a mimimum expansion to a model of* $\Delta^1_1\text{-CA}_0$. *And if* $\mathcal{M}$ *is not rec. sat. then* $\mathcal{M}$ *has no expansion to a model of* $\Delta^1_1\text{-CA}_0$.
- Contrast with the following results pertaining to *the standard model* $\mathbb{N} = (\omega, +, \cdot)$ *of* PA. In what follows HYP = the set of subsets of ω that are Turing reducible to the α-th jump of zero, for some ordinal $\alpha < \omega_1^{\mathrm{CK}}$.

  Theorem 1. (Kleene, 1955).
  (a) HYP = The Δ^1_1 definable subsets of $\mathbb{N}$.
  (b) $(\mathbb{N}, \mathrm{HYP})$ is the minimum model of Δ^1_1-CA.
  Theorem 2. (Gandy-Kreisel-Tait, 1962) *Let*

$$\mathfrak{X}_T = \cap\{\mathfrak{X} : (\mathbb{N}, \mathfrak{X}) \models T\},$$

  where T *is an* Π^1_1*-definable* $\mathcal{L}_2$*-theory which includes* $\Delta^1_1\text{-CA}_0$.
  Then $\mathfrak{X}_T = \mathrm{HYP}$.

Back to the Barwise-Schlipf Theorem

- **Theorem** (Barwise-Schlipf) *The following are equivalent for a nonstandard model* $\mathcal{M}$ *of* PA (*of any cardinality*).

  (**1**) $\mathcal{M}$ *is recursively saturated.*

  (**2**) *There is* $\mathfrak{X} \subseteq \mathcal{P}(M)$ *such that* $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{-}\mathrm{CA}_0$.

  (**3**) $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \Delta^1_1\text{-}\mathrm{CA}_0 + \Sigma^1_1\text{-}\mathrm{AC}.$
- The Barwise-Schlip proof of (1) $\Longrightarrow$ (3) uses Admissible Set Theory, and appears to be deep.
- In an exposition of this theorem by Smoryński (JSL, 1981) a more direct proof of this implication, attributed to Feferman and Stavi (independently), is presented. This same proof is essentially repeated in Simpson's SOSOA. We will shortly see this proof.
- The implication (3) $\Longrightarrow$ (2) is of course trivial. As we shall see, the proof of the implication (2) $\Longrightarrow$ (1) given by Barwise and Schlipf, is fairly short and plausible, but has a nontrivial gap.

The Feferman-Stavi proof

Recall that $\Delta^1_1\text{-CA}$ is provable in $\Sigma^1_1\text{-AC}_0$, and that in the presence of ACA_0, $\Sigma^1_1\text{-AC}$ is equivalent to $\Sigma^1_1\text{-Coll}$.

Assuming $\mathcal{M}$ is recursively saturated, and $\mathfrak{X} = \mathrm{Def}(\mathcal{M})$, we will verify that $\Sigma^1_1\text{-Coll}$ holds in $(\mathcal{M}, \mathfrak{X})$. For this purpose, suppose for some parameter $A \in \mathfrak{X}$ we have:

(1) $(\mathcal{M}, \mathfrak{X}) \models \forall x\ \exists X\ \psi(x, X, A)$.

Let $\alpha(m, v)$ be the arithmetical formula that defines A, where $m \in M$ is a number parameter. Then

(2) $(\mathcal{M}, \mathfrak{X}) \models \forall x\ \theta(x)$, where

$$\theta(x) := \bigvee_{\varphi(y,v) \in \mathrm{Form}} \exists y\ \psi(x, X/\varphi(y, v), A/\alpha(m, v)).$$

The Feferman-Stavi Proof, cont'd

We claim that (3) below holds.

(3) There is some $n \in \omega$ such that $\mathcal{M} \models \forall x \; \theta_n(x)$, where

$$\theta_n(x) := \bigvee_{\varphi(y,v) \in \mathrm{Form}_n} \exists y \; \psi(x, X/\varphi(y,v), A/\alpha(m,v)),$$

where Form_n is the set of Σ_n-arithmetical formulae.
Suppose (3) is false, then we have:

(4) $\mathcal{M} \models \exists x \; \neg\theta_n(x)$ for each $n \in \omega$.

Let $\Gamma(x) := \{\neg\theta_n(x) : n \in \omega\}$. It is easy to see that $\Gamma(x)$ is recursive. By (4), for each $n \in \omega$, $\Gamma(x)$ is finitely realizable in $\mathcal{M}$, so by recursive saturation of $\mathcal{M}$, $\Gamma(x)$ is realized in $\mathcal{M}$, i.e., $\mathcal{M} \models \exists x \; \neg\theta(x)$, which contradicts (2) and completes the verification of (3).

The Feferman-Stavi Proof, cont'd

Let $\mathfrak{X}_n := \mathrm{Def}_n(\mathcal{M})$ = parameterically Σ_n-definable subsets of $\mathcal{M}$. Note that since Σ_n-satisfaction is definable in $\mathcal{M}$, there is some $B \in \mathfrak{X}$ that codes $\mathfrak{X}_n$, i.e.,

$$\mathfrak{X}_n = \{(B)_m : m \in M\}.$$

Therefore, by (3) we have:

(5) $(\mathcal{M}, \mathfrak{X}) \models \forall x \; \exists y \; \psi(x, (B)_y, A)$.

By quantifying out B, (5) readily yields:

(6) $(\mathcal{M}, \mathfrak{X}) \models \exists Y \; \forall x \; \exists y \; \psi(x, (Y)_y, A)$.

This concludes the verification of Σ^1_1-Collection (and therefore Σ^1_1-AC) in $(\mathcal{M}, \mathfrak{X})$. $\square$

The gap (1)

Suppose $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1$-CA. Suppose $\mathcal{M}$ is not recursively saturated. Then by an overspill argument, there is no partial satisfaction class in $\mathfrak{X}$ that is correct for all standard formulae.

Suppose $\Phi(x)$ is a recursive type that is not realized. For each $m \in M$ let $\varphi_m \in \Phi(x)$ be the first formula in Φ that m does not realize.

Let $Y = \{\ulcorner \varphi_m \urcorner : m \in M\}$. Clearly $Y \subseteq \omega$, and Y is infinite (by finite satisfiability of Φ). They it is claimed that Y is Δ^1_1-definable in $(\mathcal{M}, \mathfrak{X})$, and therefore $Y \in \mathfrak{X}$, which implies that $\omega \in \mathfrak{X}$ (since Y is infinite), thus contradicting $(\mathcal{M}, \mathfrak{X}) \models \mathrm{ACA}_0$.

Here is the proposed Σ^1_1-definition, where $\mathrm{Sat}(z, X)$ expresses "X is a satisfaction predicate for formulae with length less than or equal to z".

$(\varphi \in \Phi) \wedge \exists z(z = \neg\varphi \wedge \exists x \exists X$
$[\mathrm{Sat}(z, X) \wedge (\neg\varphi, x) \in X \wedge \forall \psi < \varphi(\psi \in \Phi \to (\psi, x) \in X)])$.

The above works, i.e., it defines Y. And here is the proposed Π^1_1-definition:

$(\varphi \in \Phi) \wedge \exists z(z = \neg\varphi \wedge \exists x \forall X$
$[\mathrm{Sat}(z, X) \to (\neg\varphi, x) \in X \wedge \forall \psi < \varphi(\psi \in \Phi \to (\psi, x) \in X)]$.

A close look reveals that the above defines $Y \cup (\Phi^{\mathcal{M}} \setminus \omega)$.

The gap (2)

- The same gap is present in Murawski's (1976) account.
- Smoryński (1981) encapsulates the problematic direction of the proof of Barwise and Schlipf as the following lemma.
  Purported Lemma. *If* $\mathcal{M}$ *is not recursively saturated, and* $(\mathcal{M}, \mathfrak{X}) \models \mathrm{ACA}_0$, *then* ω *is* Δ^1_1*-definable in* $(\mathcal{M}, \mathfrak{X})$.
- In the next part of the talk we will show that the above Lemma is false by using a construction that appears in a 1987 paper (JSL) of Matt Kaufmann and Jim Schmerl, by showing:
- **Theorem** *Every completion* T *of* PA *has a nonstandard, finitely generated* (*so not recursively saturated*) *model* $\mathcal{M}$ *such that* ω *is not* Δ^1_1*-definable in* $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$.
- In the next part, we will also see how to establish the problematic direction in the Barwise-Schlipf theorem by using machinery developed by Matt Kaufmann and Jim Schmerl in a 1984 paper (APAL).

End of Part I

Recall ...

Theorem (Barwise-Schlipf, 1975) *The following are equivalent for a nonstandard model* $\mathcal{M}$ *of* PA (*of any cardinality*).

(1) $\mathcal{M}$ *is recursively saturated.*

(2) *There exists* $\mathfrak{X} \subseteq \mathcal{P}(M)$ *such that* $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{-}\mathrm{CA}_0$.

(3) $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \Delta^1_1\text{-}\mathrm{CA}_0 + \Sigma^1_1\text{-}\mathrm{AC}$.

- The Barwise-Schlip proof of (1) $\Longrightarrow$ (3) uses Admissible Set Theory, and appears to be deep.In first part of the talk we saw a proof devised by Feferman and Stavi from first principles.
- The implication (3) $\Longrightarrow$ (2) is of course trivial. In the first part of the talk we saw that the proof of the implication (2) $\Longrightarrow$ (1) given by Barwise and Schlipf has a nontrivial gap.

Proposition 1: $ACA_0 \vdash (\Sigma^1_1\text{-Coll} \to \Sigma^1_1\text{-AC})$.

Proof. Suppose $(\mathcal{M}, \mathcal{X}) \models ACA_0$ and $\forall x\, \exists X\, \varphi(x, X)$. Then by Σ^1_1-Coll:

(1) $(\mathcal{M}, \mathcal{X}) \models \exists Y\, \forall x\, \exists y\, \varphi(x, (Y)_x)$.

Let $B \in \mathcal{X}$ such that:

(2) $(\mathcal{M}, B) \models \forall x\, \exists y\, \varphi(x, (B)_y)$.

Consider $f \in \mathcal{X}$ given by:

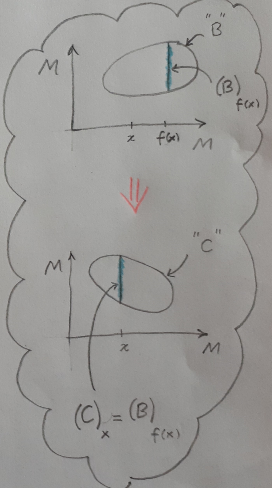
$$m \xmapsto{\ f\ } \mu y\, \varphi(m, (B)_y).$$

Then:

(3) $(\mathcal{M}, B, f) \models \forall x\, \varphi(x, (B)_{f(x)})$

Let $C = \{\langle m, y\rangle : y \in (B)_{f(m)}\} \in \mathcal{X}$.

(4) $(\mathcal{M}, C) \models \forall x\, \varphi(x, (C)_x)$.



Loose ends from the last talk (2)

Proposition 2 $\quad ACA_0 \vdash \Sigma^1_1\text{-}AC \longrightarrow \Delta^1_1\text{-}CA$.

Proof. Suppose $(M, \mathcal{X}) \models \Sigma^1_1\text{-}AC$, with:

(1) $A = \{ m \in M : (M, \mathcal{X}) \models \overbrace{\exists X\, \varphi^+(X, m)}^{\sigma^+(m)} \}$;

(2) $M \setminus A = \{ m \in M : (M, \mathcal{X}) \models \underbrace{\exists X\, \varphi^-(X, m)}_{\sigma^-(m)} \}$.

$X_E = \{ m : 2m \in X \}$

$X_O = \{ m : 2m+1 \in X \}$

Clearly:

(3) $(M, \mathcal{X}) \models \forall m\, (\sigma^+(m) \vee \sigma^-(m))$.

By coding:

(4) $(M, \mathcal{X}) \models \forall m\, \exists X\, (\varphi^+((X)_E, m) \vee \varphi^-((X)_O, m))$

By $\Sigma^1_1\text{-}AC$ there is $B \in \mathcal{X}$ such that

(5) $(M, B) \models \forall m\, \varphi^+((B)_E, m) \vee \varphi^-((B)_O, m))$.

Therefore $\quad A = \{ m \in M : \varphi^+((B)_E, m) \} \in \mathcal{X}$.

Recasting Σ^1_1-definability (1)

Definition. Suppose that $\mathcal{M} \models \mathrm{PA}$ and $A \subseteq M$. Then, A is *recursively σ-definable* if there is a recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$ of formulas, each $\varphi_n(x)$ defining a subset $A_n \subseteq M$, such that $A = \bigcup_{n<\omega} A_n$. More precisely, for such a sequence to be recursive, it is necessary that there is a finite set $F \subseteq M$ such that any parameter occurring in any $\varphi_n(x)$ is in F, so technically the definition requires the existence of a witnessing recursive sequence $\langle \varphi_n(x, \overline{y}) : n < \omega \rangle$ of formulas, and some choice of parameters $\overline{m} \in M$.

Recasting Lemma. *Suppose that $\mathcal{M} \models \mathrm{PA}$ and $A \subseteq M$.*

(**a**) *If A is Σ^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$, then A is recursively σ-definable.*

(**b**) *If $\mathcal{M}$ is not recursively saturated, $\mathrm{Def}(\mathcal{M}) \subseteq \mathfrak{X} \subseteq \mathcal{P}(M)$ and A is recursively σ-definable, then A is Σ^1_1-definable in $(\mathcal{M}, \mathfrak{X})$.*

Recasting Σ_1^1-definability (2)

Proof.

- **(a)** Suppose that A is Σ_1^1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$ by the formula $\exists X\ \theta(x, X)$. Let $\varphi_n(x)$ be the formula asserting: there is a Σ_n-definable subset X such that $\theta(x, X)$. Then $\langle \varphi_n(x) : n < \omega \rangle$ is recursive and shows that A is recursively σ-definable.
- **(b)** Recall that $\mathrm{Sat}(x, X)$ is the formula asserting that X is a satisfaction class for all formulas of length at most x. *Since $\mathcal{M}$ is assumed in this part not to be recursively saturated, there is no $X \subseteq M$, and no nonstandard $m \in M$ such that*:

$$(\mathcal{M}, X) \models \mathrm{PA}^* \text{ and } (\mathcal{M}, X) \models \mathrm{Sat}(m, X).$$

Recasting Σ^1_1-definability (3)

- **(b), cont'd.** Let A be recursively σ-definable by the recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$. We can assume that $\ell(\varphi_n(x)) < \ell(\varphi_{n+1}(x))$ for all $n < \omega$, where $\ell(\varphi(x))$ is the length of $\varphi(x)$ (by replacing $\varphi_n(x)$ with $\bigvee_{i \leq n} \varphi_i(x)$). The sequence $\langle \varphi_n(x) : n < \omega \rangle$ is coded in $\mathcal{M}$, so let $d \in M$ be nonstandard such that $\langle \varphi_n(x) : n < d \rangle$ extends $\langle \varphi_n(x) : n < \omega \rangle$ and $\ell(\varphi_n(x))$ is standard iff n is. Then A is Σ^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$ by the formula $\exists X \theta(x, X)$, where

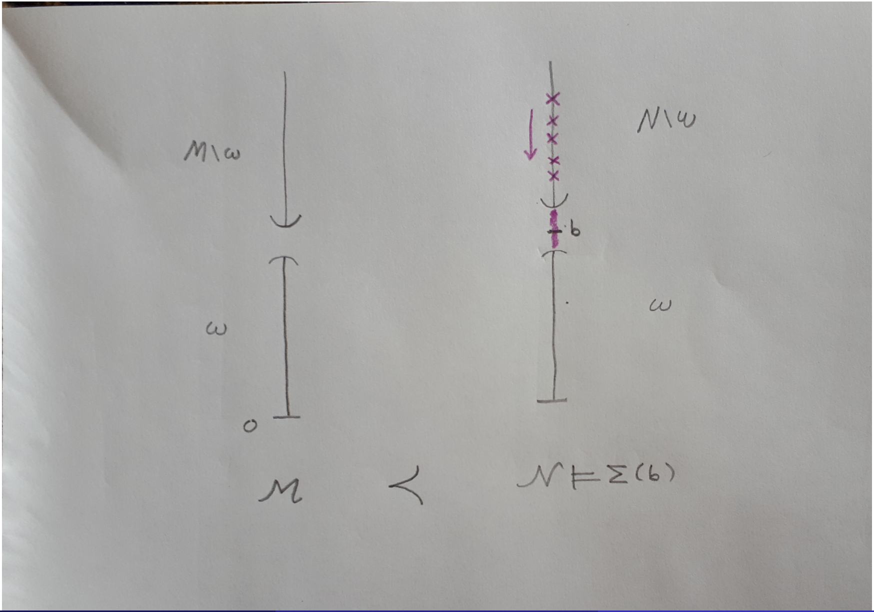
$$\theta(x, X) = \exists z [\mathrm{Sat}(z, X) \wedge \exists n < d \big(\ell(\varphi_n) \leq z \wedge \langle \varphi_n, x \rangle \in X\big)].$$

Thus, A is Σ^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$. The same definition works in $(\mathcal{M}, \mathfrak{X})$. $\square$

The gap in the Barwise-Schlipf proof is real (1)

- **Definition.** If I is a cut of $\mathcal{M}$, then we say that I is *definable* if there is some finitely realizable type $\Sigma(x)$ over $\mathcal{M}$ (where $\Sigma(x)$ uses at most finitely many parameters from M), such that if $\mathcal{M} \prec \mathcal{N}$ and $b \in N$ realizes $\Sigma(x)$, then $\mathcal{N}$ fills I with b. Moreover, I is *recursively definable* if $\Sigma(x)$ is recursive.
- **First Kaufmann-Schmerl Theorem.** *The minimal model $\mathcal{M}_T$ of every consistent completion T of PA has a simple nonstandard extension in which ω is not recursively definable.*
- The above theorem appears as Corollary 2.8 of the following paper: Matt Kaufmann and James H. Schmerl, *Remarks on weak notions of saturation in models of Peano arithmetic*, **J. Symbolic Logic,** 52 (1987), 129–148.

Filling a gap



The gap in the Barwise-Schlipf proof is real (2)

- **Theorem.** *Every completion T of* PA *has a nonstandard, finitely generated (hence not recursively saturated) model $\mathcal{M}$ such that ω is not Δ^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$.*
- **Proof**. Let T be a completion of PA. By the first Kaufmann-Schmerl Theorem, there is a finitely generated $\mathcal{M} \models T$ in which ω is not recursively definable. Therefore $M \setminus \omega$ is not recursively σ-definable in $\mathcal{M}$. So by part (a) of Recasting Lemma, ω is not Π^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$. $\square$.
- **Remark.** If $\mathcal{M}$ is a short recursively saturated model of T that is not tall (and therefore is not recursively saturated), then by the same reasoning as above ω is not Δ^1_1-definable in $(\mathcal{M}, \mathrm{Def}(\mathcal{M}))$.

The Second Kaufmann-Schmerl Theorem

- **Definition** (interval type). An *interval type* $\Gamma(v, \overline{m})$ over a model $\mathcal{M}$ of PA is a type over $\mathcal{M}$ (with finitely many parameters $\overline{m}$ from M) such that every formula in Γ is of the form $\tau_1(\overline{m}) \leq v \leq \tau_2(\overline{m})$, for some pair of terms $\tau_1(\overline{y})$ and $\tau_2(\overline{y})$, and whenever γ_1, $\gamma_2 \in \Gamma$, then either $\mathcal{M} \models \gamma_1 \to \gamma_2$, or $\mathcal{M} \models \gamma_2 \to \gamma_1$.
- **The Second Kaufmann-Schmerl Theorem.** *The realizability of every short finitely realizable type $\Sigma(v, \overline{a})$ over a model $\mathcal{M}$ of PA can be "effectively reduced" to the realizability of an interval type $\Gamma(v, \overline{a}, d)$ over $\mathcal{M}$ in the following sense*:
  (a) *$\Gamma(v, \overline{m}, d)$ is finitely realizable in $\mathcal{M}$ for every nonstandard $d \in M$; and if for some (nonstandard) $d \in M$, $\Gamma(v, \overline{m}, d)$ is realized in $\mathcal{M}$, then $\Sigma(v, \overline{a})$ is realized in $\mathcal{M}$.*
  (b) *$\Gamma(v, \overline{y}, z)$ is recursive in $\Sigma(v, \overline{y})$. In particular, if Σ is recursive, then so is Γ.*
- The above Theorem follows from Lemma 2.4 of the following paper: M. Kaufmann and J. H. Schmerl, *Saturation and simple extensions of models of Peano arithmetic*, **Ann. Pure Appl. Logic** 27 (1984),

Circumventing the Gap (1)

Theorem: *If $\mathcal{M}$ is nonstandard and $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{-}\mathrm{CA}_0$, then $\mathcal{M}$ is recursively saturated.*

Proof. We will show that if $\mathcal{M}$ is nonstandard and not recursively saturated and $\mathfrak{X} \subseteq \mathcal{P}(M)$, then $(\mathcal{M}, \mathfrak{X}) \not\models \Delta^1_1$-CA. We can assume that $(\mathcal{M}, \mathfrak{X}) \models \mathrm{ACA}_0$. There are two cases depending on whether $\mathcal{M}$ is short or tall.

Case 1: $\mathcal{M}$ *is short*: Let $c \in M$ be such that the elementary submodel of $\mathcal{M}$ generated by c is cofinal in $\mathcal{M}$. Fix a nonstandard element $e \in M$, and let $\langle \varphi_n(x) : n < \omega \rangle$ be a recursive sequence of formulas (with c and e as the only parameters) such that $\varphi_n(x)$ defines $d_n \in M$, where d_n is the least element that is above all elements that are definable from c via a Σ_n formula of length at most e. It can be readily verified that $\langle d_n : n < \omega \rangle$ is strictly increasing, and unbounded in $\mathcal{M}$. Let $D = \{d_n : n < \omega\}$. Since $(\mathcal{M}, \mathfrak{X}) \models \mathrm{ACA}_0$, then $D \notin \mathfrak{X}$ as otherwise $\omega \in \mathfrak{X}$. Clearly, D is recursively σ-definable ; its complement also is (using the recursive sequence $\langle \psi_n(x) : n < \omega \rangle$, where $\psi_0(x)$ is $x < d_0$ and $\psi_{n+1}(x)$ is $d_n < x < d_{n+1}$).

Circumventing the Gap (2)

Case 2: $\mathcal{M}$ *is tall*: Since $\mathcal{M}$ is tall and not recursively saturated, there is a finitely realizable (in $\mathcal{M}$) recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$ of formulas, among which is a formula $x < b$, which is not realizable in $\mathcal{M}$. By the second Kaufmann-Schmerl theorem, we can assume that each $\varphi_n(x)$ defines an interval $[a_n, b_n]$, where $a_n < a_{n+1} < b_{n+1} < b_n$. Then, the cut $I = \sup\{a_n : n < \omega\} = \inf\{b_n : n < \omega\}$, so both I and its complement are recursively σ-definable. By part (b) of Recasting Lemma I is Δ^1_1-definable in $(\mathcal{M}, \mathfrak{X})$. Since $I \notin \mathfrak{X}$, then $(\mathcal{M}, \mathfrak{X}) \not\models \Delta^1_1$-CA. □

Proof of the First Schmerl-Kaufmann Theorem (1)

We first prove the following.

Preliminary Theorem. *Let* T_0 *be any consistent extension of* PA *which represents itself. Then* T_0 *has a consistent completion such that* ω *is not recursively definable in* $\mathcal{M}_T$.

Proof. Enumerate all recursive types (and assume they are closed under conjunction) as $\Sigma_n(x)$ for $n \in \omega$ (no need to worry about parameters since we will be looking at types over $\mathcal{M}_T$). T will be built as the union of consistent theories T_n. Let c_θ be the term denoting the least number satisfying $\theta(x)$ (and otherwise equal to 0 if there is no number x satisfying $\theta(x)$).

Suppose T_n has been constructed and $n \geq 0$. Let $\Sigma(x)$ denote $\Sigma_n(x)$, assume that $T_n \cup \Sigma$ is consistent Then we will build T_{n+1} such that one of the following two conditions hold:

(1) $T_{n+1} = T_n \cup \{\forall x\,(\sigma(x) \rightarrow x < k)\}$ for some $\sigma(x) \in \Sigma(x)$ and some $k \in \omega$.

(2) $T_{n+1} = T_n \cup \{\exists x\,(\sigma(x) \wedge x \geq c_\theta) : \sigma(x) \in \Sigma(x)\}$, for some θ such that $T_n \vdash c_\theta > k$ for all $k \in \omega$.

Proof of the First Schmerl-Kaufmann Theorem (2)

Case 1. There is a choice of $\sigma \in \Sigma$ and $k \in \omega$ such that T_{n+1} as in (1) is consistent, which makes our choice of T_{n+1} clear.
Case 2. Case 1 fails. Let $\theta(x)$ be a fixed-point for the formula $\mathrm{Prov}_{T_n \cup \Sigma(v)}(x, \ulcorner v < c_\theta \urcorner)$, i.e.,

$$T_n \vdash \theta(x) \leftrightarrow \mathrm{Prov}_{T_n \cup \Sigma(v)}(x, \ulcorner v < c_\theta \urcorner).$$

Claim: $T_n \vdash c_\theta > k$ for all $k \in \omega$. If not, there is a consistent finite extension T_n^+ of T_n and some $k \in \omega$ such that $T_n^+ \vdash c_\theta = k$, i.e., $T_n^+ \vdash \theta(k)$. Therefore $T_n^+ \vdash \mathrm{Prov}_{T_n \cup \Sigma(v)}(k, \ulcorner v < c_\theta \urcorner)$, which in turn implies that $T_n \cup \Sigma(x) \vdash x < c_\theta$, so $T_n^+ \cup \Sigma(x) \vdash x < c_\theta$. Hence there is some $\sigma(x) \in \Sigma(x)$ such that $T_n^+ \vdash \forall x\, (\sigma(x) \to x < k)$, which contradicts our assumption that Case I fails, and completes the proof of the claim about c_θ. It is not hard to see that $T_n \cup \Sigma(x) \cup \{x \geq c_\theta\}$ is consistent, and therefore the choice of $T_{n+1} = T_n \cup \{\exists x\, (\sigma(x) \wedge x \geq c_\theta) : \sigma(x) \in \Sigma(x)\}$ results in a consistent theory. □

Proof of the First Schmerl-Kaufmann Theorem (3)

- **First Kaufmann-Schmerl Theorem.** *The minimal model $\mathcal{M}_T$ of every consistent completion T of PA has a simple nonstandard extension in which such that ω is not recursively definable.*
- **Proof.** Add a new constant c to the language of arithmetic and apply the previous theorem to the theory:

$$T^+ = T \cup \{\varphi \longleftrightarrow [(c)_{\ulcorner\varphi\urcorner} = 0] : \varphi \in \mathsf{Sent}_{\mathrm{PA}}\},$$

where $(c)_n$ is the exponent of the n-th prime in the prime decomposition of c. By design, T^+ represents T. $\square$

Proof of the second Kaufmann-Schmerl Theorem (1)

Definition (in PA). Suppose $[a, b]$ is an interval and X is a finite set.
$f : [a, b] \longrightarrow_{k\text{-onto}} X$ is defined by induction on k as follows:
(Base) $f : [a, b] \longrightarrow_{0\text{-onto}} X$ means $f : [a, b] \longrightarrow_{\text{onto}} X$.
(Inductive) $f : [a, b] \longrightarrow_{n+1\text{-onto}} X$ means $\forall Y \subseteq X\ \exists [c, d] \subseteq [a, b]$ such that $f : [c, d] \longrightarrow_{n\text{-onto}} Y$.

Lemma (in PA). *For all numbers k and all finite sets X there is an interval $[a, b]$ and a function $f : [a, b] \longrightarrow_{k\text{-onto}} X$.*
Proof. Induction on k. Case $k = 0$ is clear. For the inductive case suppose $k = n$, and X is some finite set. For each $Y \subseteq X$ by inductive assumption there is $[a_Y, b_Y] \subseteq [a, b]$ and f_Y such that

$$f_Y : [a_Y, b_Y] \longrightarrow_{n\text{-onto}} Y.$$

WLOG we can arrange $[a_Y, b_Y] \cap [a_Z, b_Z] = \varnothing$ if $Y \neq Z$.
Let $a = \min\{a_Y : Y \subseteq X\}$, $b = \max\{a_Y : Y \subseteq X\}$, and let $g : [a, b] \to X$ be any extension of $\cup \{f_Y : Y \subseteq X\}$. g is clearly $(n+1)$-onto. □

Proof of the second Kaufmann-Schmerl Theorem (2)

Lemma (Effectively coding short types by interval types). *Given a short type* $\{\sigma_n(v, \overline{y}) : n \in \omega\}$ *such that* $\sigma_0 = \{v < y_0\}$*, there is an interval type*

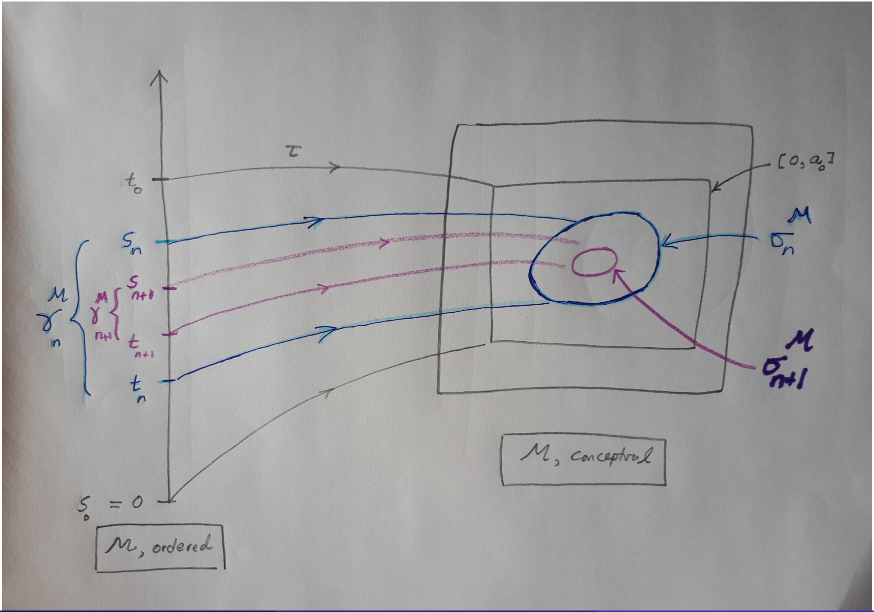
$$\Gamma = \{\gamma_n(v, \overline{y}, z) : n \in \omega\},$$

together with a term $\tau(x, y_0, z)$*, such that* Γ *is recursive in* Σ*, and for all* $\mathcal{M} \models \mathrm{PA}$*, and all* $\overline{a} \in M$ *the following hold*:

(i) *If* $\Sigma(v, \overline{a})$ *is finitely realizable in* $\mathcal{M}$*, then for every nonstandard* $d \in M$*,* $\Gamma(v, \overline{a}, d)$ *is finitely realizable in* $\mathcal{M}$*.*

(ii) *If* $\Gamma(v, \overline{a}, d)$ *is realized in* $\mathcal{M}$ *for some (nonstandard)* d*, then* $\Sigma(v, \overline{a})$ *is realized in* $\mathcal{M}$*.*

Proof of the second Kaufmann-Schmerl Theorem (3)



Proof of the second Kaufmann-Schmerl Theorem (4)

Proof.
Choose τ, s_0, t_0 such that the following is PA-provable:

$$\tau(., y_0, z) : [s_0(\overline{y}, z), t_0(\overline{y}, z)] \longrightarrow_{z\text{-onto}} [0, y_0].$$

Generally choose s_{n+1}, t_{n+1} so that the following conjunction is PA-provable:

$$(z > n) \to s_n(\overline{y}, z) \leq s_{n+1}(\overline{y}, z) \leq t_{n+1}(\overline{y}, z) \leq t_n(\overline{y}, z)$$

$$\wedge$$

$$\tau(., y_0, z) \upharpoonright [s_{n+1}(\overline{y}, z), t_{n+1}(\overline{y}, z)] \longrightarrow_{(z-n-1)\text{-onto}} \{x \leq y_0 : \sigma_n(x, y)\}.$$

Then choose:

$$\gamma_n = (z > n) \wedge s_{n+1}(\overline{y}, z) \leq v \leq t_{n+1}(\overline{y}, z).$$

$\square$

Envoi

- **Theorem.** *Suppose* $\mathcal{M} \models \mathrm{PA}$.
- (a) (Kaufmann-Schmerl) *$\mathcal{M}$ has no definable cuts iff $\mathcal{M}$ is ω-saturated.*
- (b) (Kaufmann-Schmerl) *$\mathcal{M}$ has no recursive definable cuts iff $\mathcal{M}$ is recursively saturated.*
- (c) (Pabion-Richard) *For any uncountable cardinal κ, $(M, <^{\mathcal{M}})$ is κ-saturated iff $\mathcal{M}$ is κ-saturated.*
- **Remarks**
- Kaufmann and Schmerl gave an alternative proof for (c) above, and this new proof makes it clear that for any uncountable cardinal κ, a model $\mathcal{M}$ of ZFC is κ-saturated iff $(\mathrm{Ord}, \in)^{\mathcal{M}}$ is κ-saturated.
- Tarski's elimination of quantifiers for real closed fields can be used to show that (c) above also holds for real closed fields, even for $\kappa = \omega$.
- The analogue of (c) for Presburger arithmetic is known to be false.

Thank you for your attention