Distributivity and base trees for $P(\kappa)/ < \kappa$

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Trees of maximal antichains in $P(\kappa)/<\kappa$

- Growing downward; nodes are elements of $[\kappa]^\kappa$
- Each level is a maximal antichain refining the levels above it
- A **distributivity tree** is one where there is no maximal antichain refining all levels. Sometimes called a refining matrix in the literature
- A **base tree** is a distributivity tree T where for every $x \in [\kappa]^\kappa$, there exists $t \in T$ with $t \subseteq^* x$. Sometimes called a base matrix in the literature

Definitions, observations, classical results

- $\mathfrak{a}_\kappa$ is the minimal $\lambda > \kappa$ such that there exists a λ-sized maximal antichain in $P(\kappa)/<\kappa$

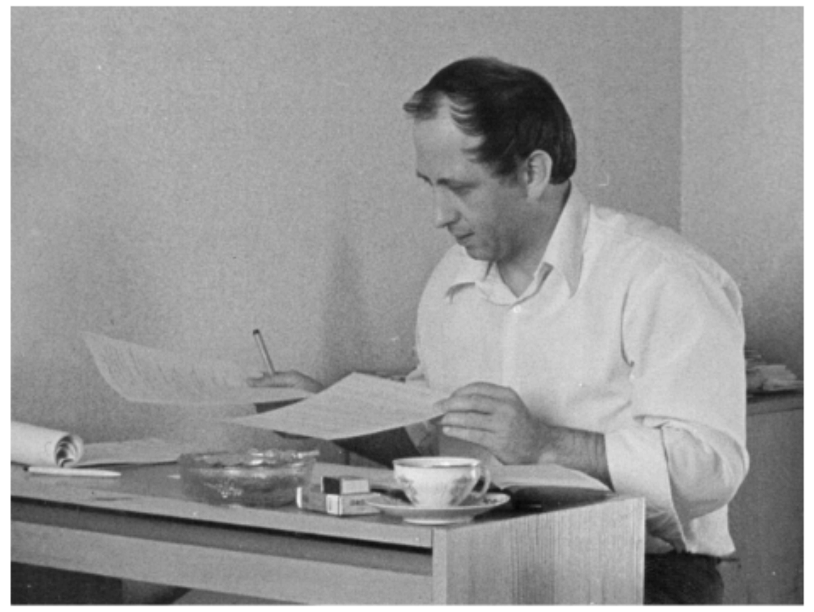
- $\mathbb{P}$ is κ-distributive if it doesn't add a new κ-sequence of ordinals

  - $\mathfrak{h}(\mathbb{P})$ is the least κ such that $\mathbb{P}$ is not κ-distributive. $\mathfrak{h}(P(\omega)/\mathrm{fin}) = \mathfrak{h}$.

- $\mathfrak{h}$ is the minimal height of a distributivity tree for $P(\omega)/\mathrm{fin}$

  - (1980; Balcar, Pelant, Simon): There is a base tree for $P(\omega)/\mathrm{fin}$ of height $\mathfrak{h}$

  - (1972; Balcar, Vopěnka): For $\mathrm{cf}(\kappa) > \omega$ there is a distributivity tree of height ω. For $\mathrm{cf}(\kappa) = \omega$ there is a distributivity tree of height ω_1



Bohuslav Balcar

B. Balcar and P. Vopěnka. On systems of almost disjoint sets. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 20(6) (1972), 421–424

B. Balcar, J. Pelant, P. Simon. The space of ultrafilter on ℕ covered by nowhere dense sets. *Fund. Math.* 110(1) (1980), 11–24

Prior observations and recent questions

- (2016):
  - A κ-Aronszajn tree can be used to build distributivity trees of height κ in $P(\kappa)/<\kappa$ (i.e. "for κ")
  - Consistently for $\mathrm{cf}(\kappa) > \omega$ there is a base tree of height ω
  - Consistently for $\mathrm{cf}(\kappa) = \omega$ there is a base tree of height ω_1
- (2021; Fischer, Koelbing, Wohofsky):
  1. Can there exist a base tree of height $> \mathfrak{h}$ for ω?
  2. Does/can there exist a distributivity tree of intermediate height $\mu \in (\omega, \kappa)$ for κ?
  3. Does/can there exist a distributivity tree of height $> \kappa$ for κ?

V. Fischer, M. Koelbing, W. Wohofsky. The distributivity spectrum of $P(\omega)$/fin. *Preprint.* https://www.logic.univie.ac.at/~vfischer/distributivity07.pdf (2022)

G. Galgon. Trees, Refining, and Combinatorial Characteristics. *PhD Thesis. University of California, Irvine* (1980)

Recent answers (Question 1)

Question 1:

- (2023, Brendle): If $\mathfrak{c}$ is regular, there is a base tree of height $\mathfrak{c}$ for ω. The Cohen and Random models have base trees of all regular uncountable heights $\leq \mathfrak{c}$.

- (2023; Fischer, Koelbing, Wohofsky): There is a c.c.c. iterated forcing to add a distributivity tree of height $> \mathfrak{h}$ for ω.

J. Brendle. Base matrices of various heights. *Canad. Math. Bull.* 66 (2023), no. 4, 1237–1243

V. Fischer, M. Koelbing, W. Wohofsky. Refining systems of mad families. *Israel J. Math.* 262 (2024), 191–234

More recent answers (Questions 2 and 3)

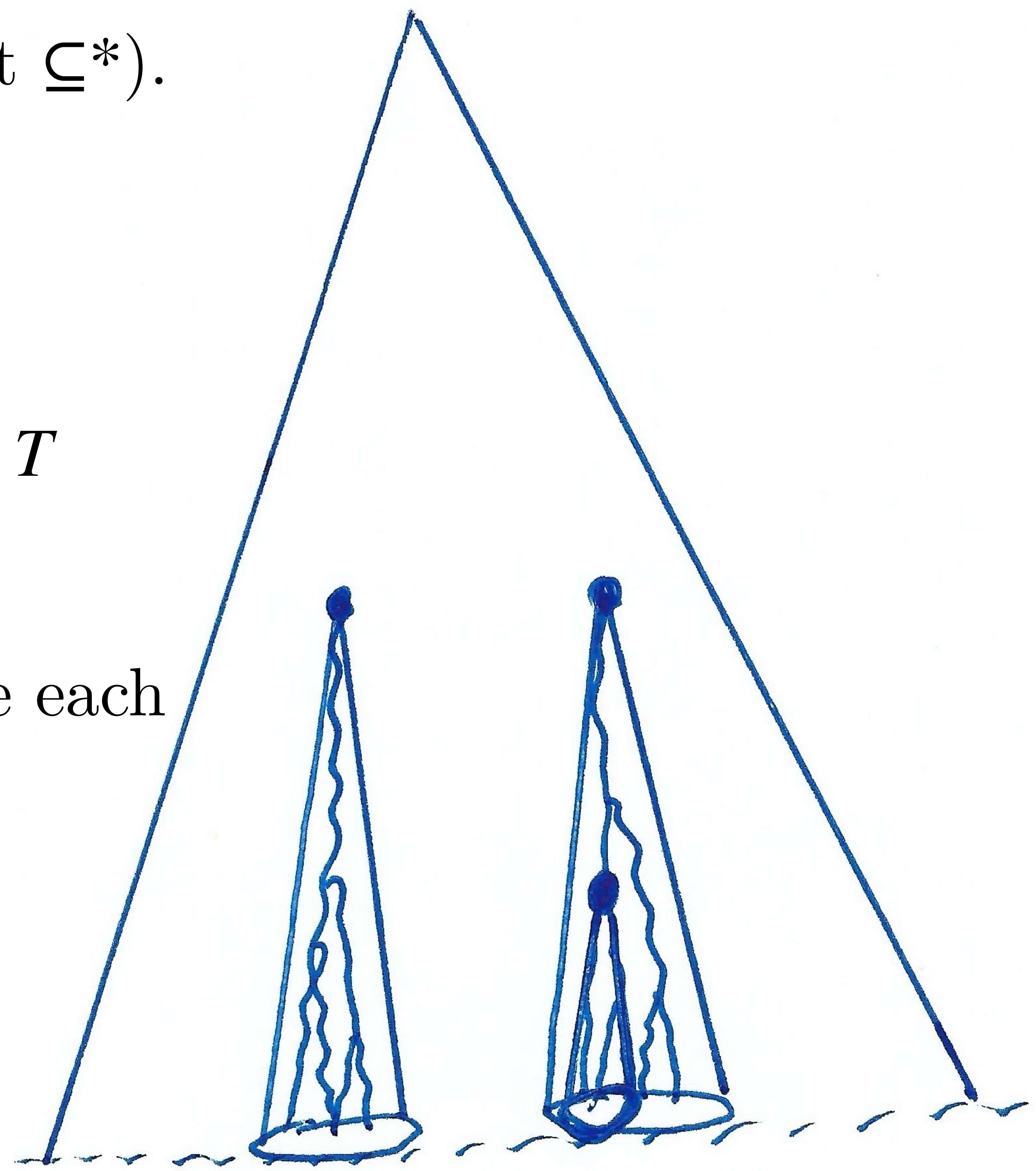
- Question 2:
  - The existence of a "partition-type" distributivity tree of height μ for κ is equivalent to the existence of a certain type of (weak) Kurepa tree.
  - If $\kappa > \omega$ is regular and $\mathfrak{a}_\kappa = 2^\kappa$, for every $\mu \in [\omega, \kappa)$ there exists a base tree of height μ for κ.
- Question 3:
  - If $\kappa > \omega$ is regular and $\mathfrak{a}_\kappa = 2^\kappa$, there exists a base tree of height κ for κ.
  - If $\kappa > \omega$ is regular, $\mathfrak{a}_\kappa = 2^\kappa$, and λ is the (regular) length of a tower in $P(\kappa)/ < \kappa$ or the limit of such cardinalities, there exists a base tree of height λ for κ.

G. Galgon. Distributivity and base trees for $P(\kappa)/ < \kappa$. *Canad. Math. Bull.* Published online (2024), 1–12

Narrow, short, distributivity trees

- A *partition-type* distributivity tree for κ is one where each level of the tree is a partition of κ (therefore of size less than κ) and the tree relation is $\subseteq$ (not $\subseteq^*$).

Example: Suppose the CH holds and consider the full binary tree T on ω_1.



  - Take ω_2-many branches through T sufficient to generate the tree and associate them with the ordinals in ω_2. We may assume every node in T has ω_2-many branches through it

  - Form the partition-type distributivity tree T' for ω_2 of height ω_1 where each partition element is the collection of branches (ordinals) inside the downward cone of the corresponding node in T.

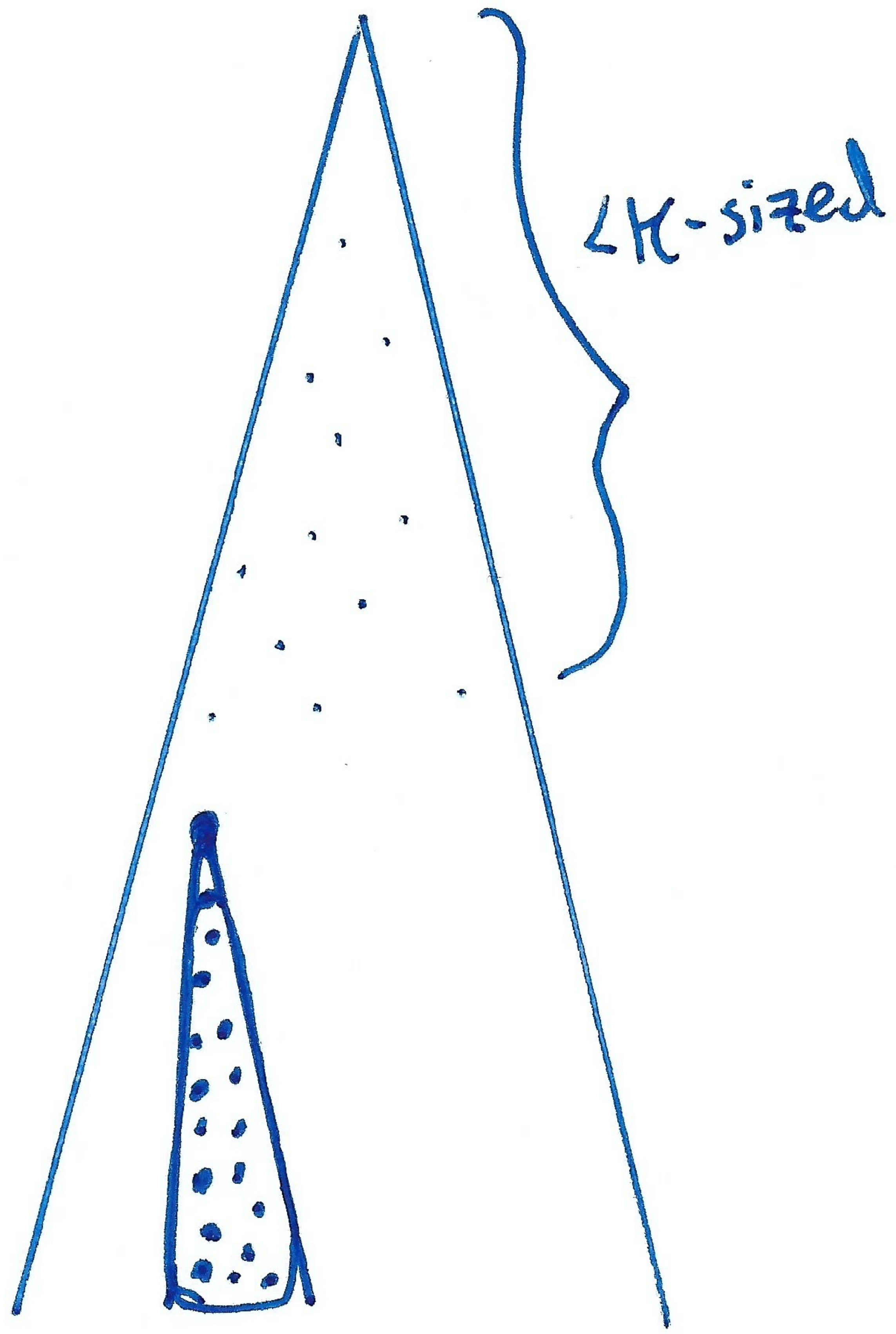
    - Levels of T' are of size ω_1, partition ω_2, and the height of T' is ω_1.

    - The intersections of the relevant subsets of ω_2 along branches are singletons and so cannot be extended. So it's a distributivity tree.

Tallest narrow distributivity tree: κ-Aronszajn

- A distributivity tree of height κ where the levels are maximal antichains of size less than κ is necessarily a κ-Aronszajn tree, because there are no towers of length κ in $P(\kappa)/<\kappa$.

- On the other hand, we may associate the nodes in a κ-Aronszajn tree with the ordinals of κ and observe that the downward nodal cones of elements on each level are (modulo $P_\kappa\kappa$) partitions of κ into fewer than κ-many pieces.
    - The resulting tree of maximal antichains must have no branches and so is in fact a distributivity tree.



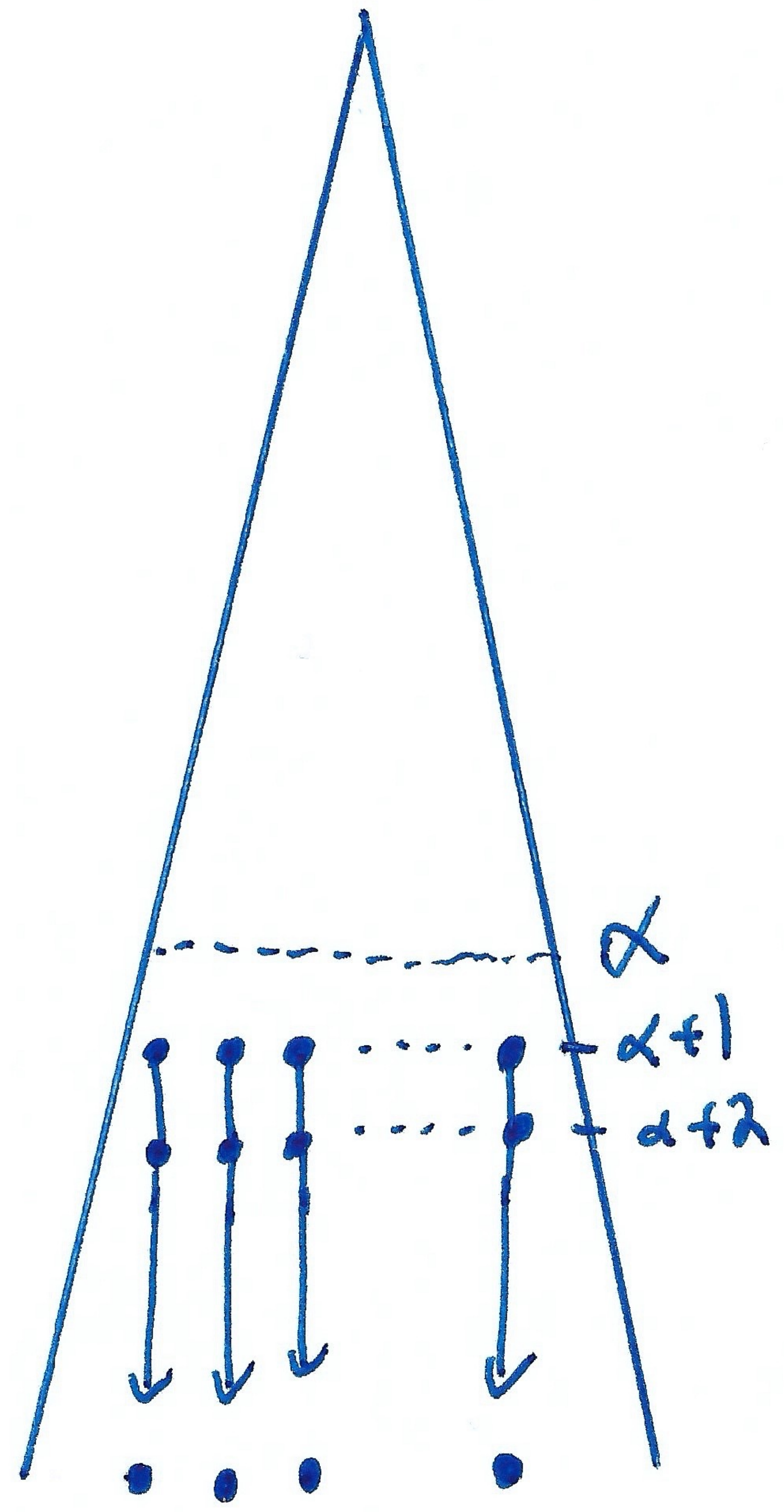
Lemma 1: No tall narrow distributivity trees

- If κ is regular and there exists $\mu < \kappa$ and a pruned tree T of height κ with $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha < \kappa$, then T is eventually nonsplitting.

- That is, there exists $\alpha < \kappa$ such that for all $\beta \in (\alpha, \kappa)$, every $s \in \mathrm{Lev}_\beta(T)$ is not splittable (all extensions of s are compatible).

  - Example: There does not exist a distributivity tree for ω_1 of height ω_2 with countable levels.

- This is proven by looking at a regressive function on a stationary subset of κ.



Lemma 2: No Aronszajn subtrees

- Similar reasoning to Lemma 1 shows $T^{\kappa}_{<\mu} \subseteq {}^{<\kappa}\kappa$, the tree consisting of $<\kappa$ -sequences in κ with fewer than μ-many nonzero values, for $\mu < \kappa$ regular, does not have any κ-Aronszajn subtrees.
  - The applications in what follows are for the $\mu = \omega$ case.

Question 2: Short, wide, base trees (1)

Preliminary definition: For $x, y \in [\kappa]^\kappa$, x is *discontinuous (everywhere) relative to* y if for every $\beta \in \lim(\kappa)$, $x(\beta) > \min(y \backslash \sup\{x(\xi) : \xi < \beta\})$

- x is *almost everywhere discontinuous relative to* y when for some $\gamma < \kappa$, this holds for every limit $\beta \in (\gamma, \kappa)$.
  - If $x \subseteq y$, saying x is everywhere discontinuous relative to y is equivalent to saying for every $\beta \in \kappa$, $x(\beta) > y(\beta)$.
  - That is, for the inverse enumerating functions for x and y, we have $f_x^{-1}(\alpha) < f_y^{-1}(\alpha)$ for every $\alpha \in x$.
  - Example: if $y \in [\kappa]^\kappa$ then the set of successor ordinals in its order topology is everywhere discontinuous relative to y.
- As long as $\mathrm{cf}(\kappa) > \omega$, there can be no $(\omega + 1)$-length $\subseteq^*$-descending sequence of elements in $[\kappa]^\kappa$ each of which is almost everywhere discontinuous relative to its predecessors, as this would yield an infinite descending sequence of ordinals.

Short, wide, base trees (2)

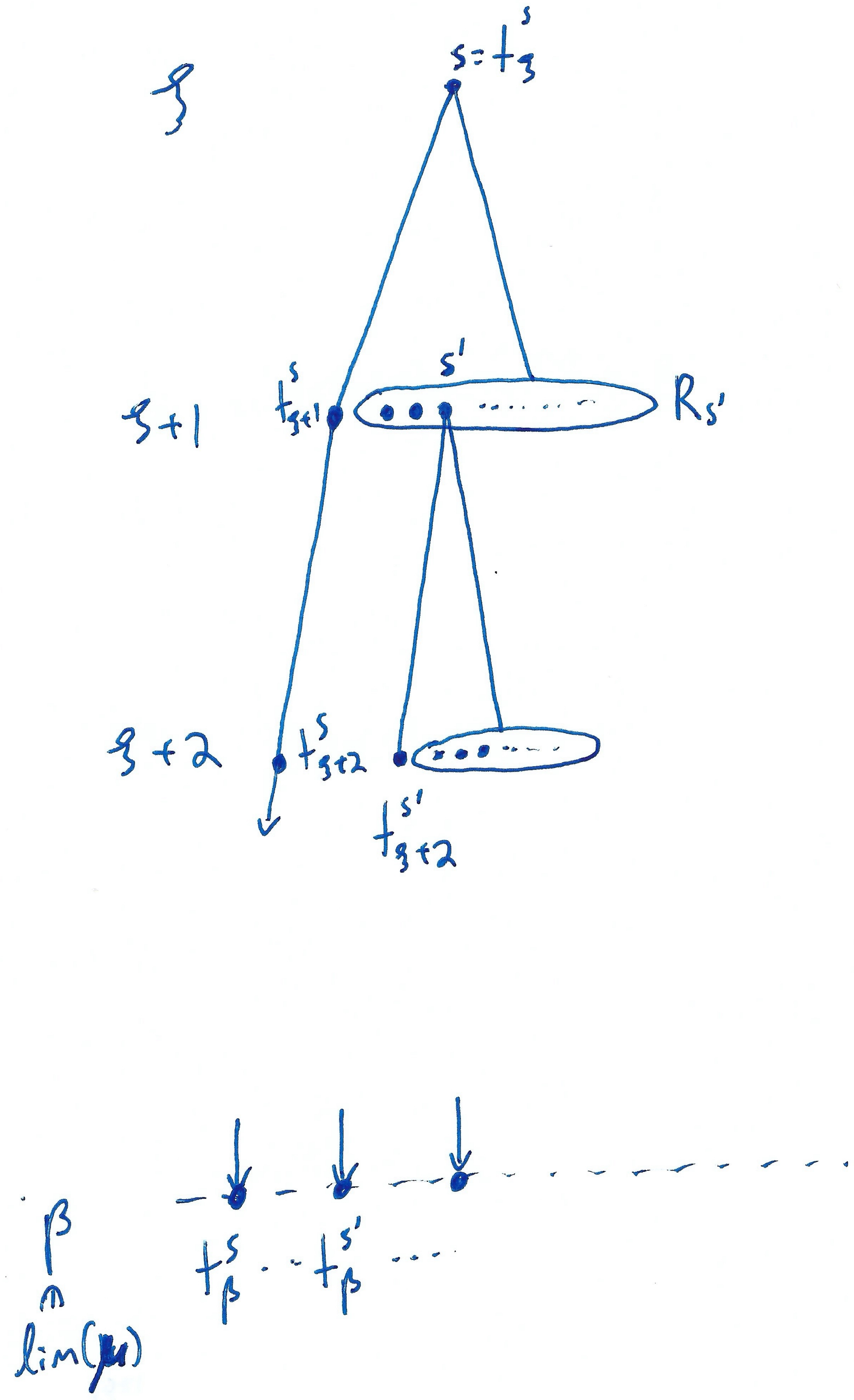
Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. For $\omega \leq \mu < \kappa$, there exists a base tree of height μ for κ with levels of cardinality 2^κ.

- The tree is built iteratively, with $\subseteq$ as the tree relation.
- There are two types of nodes—"root" nodes and "tower" nodes
  - Tower nodes are associated with μ-towers (strictly decreasing, continuous, empty intersection) through the tree. Limit levels consist entirely of tower nodes.
  - Root nodes are associated with a 2^κ-sized "root node family" of which they're a part.
  - Root nodes only occur at successor levels and are everywhere discontinuous with respect to the relevant $z \in [\kappa]^\kappa$ (which is a subset of the predecessor on the previous level).
    - So at limit levels of the tree only paths containing finitely-many root nodes have nonempty intersection and they must eventually travel along a μ-tower.
  - Diagonalization occurs against root node families in subsequent levels to ensure the base property.

Short, wide, base trees (3)

Diagonalization step:

- Imagine s' is a root node on level $\xi + 1$, below the tower node $s = t^s_\xi$ on level ξ, part of the root node family $R_{s'}$.
  - $R_{s'}$ is maximal almost disjoint of cardinality 2^κ in $z = t^s_\xi \setminus t^s_{\xi+1}$ and every element in $R_{s'}$ is everywhere discontinuous relative to z
- Let $X = \{x \in [z]^\kappa : |\{r \in R_{s'} : |x \cap r| = \kappa\}| = 2^\kappa\}$.
  - Add at least one tower inside every such x (note $|X| \leq 2^\kappa$) below a suitable s', starting with element $t^{s'}_{\xi+2} \subseteq x \cap s'$ and so on.
    - Ensure $|s' \setminus t^{s'}_{\xi+2}| = \kappa$ and split $z' = s' \setminus t^{s'}_{\xi+2}$ into another 2^κ -sized root node family.



Short, wide, base trees (5)

Maximality of levels:

- Clear for levels $\xi + 1$ as MAD families are added below every node from level ξ.
- So for $\xi \in \lim(\mu)$, need to see $\mathrm{Lev}_\xi(T)$ is a maximal antichain
  - Suppose for every $\nu < \xi$, $\mathrm{Lev}_\nu(T)$ is maximal and let $x \in [\kappa]^\kappa$
  - $\mathfrak{a}_\kappa = 2^\kappa$, so $x \in [\kappa]^\kappa$ hits (intersects in a set of size κ) either fewer than κ-many elements on each level or there is a minimal $\eta < \xi$ where x hits 2^κ-many nodes in $\mathrm{Lev}_\eta(T)$.
    - In the former case, argue $T_\xi \restriction x$ is essentially a partition-type tree of maximal antichains and there will be a κ-sized subset of x hitting one of the tower nodes on level ξ.
    - In the latter case, one notes that η cannot be a limit (as there earn't enough nonempty branches through $T_\eta \restriction x$) and then observes that 2^κ-many nodes within a particular root node family hit x on level η.
    - By the diagonalization step, a tower is then added below x starting at level $\eta + 1$. So the limit node of that tower at level ξ is a subset of x.

Short, wide, base trees (6)

Distributivity tree

- All branches through T eventually travel along some μ-tower, so T is a distributivity tree

Base tree

- As before, $\mathfrak{a}_\kappa = 2^\kappa$, so x hits either fewer than κ-many elements on each level or there is a minimal $\eta < \mu$ where x hits 2^κ-many nodes in $\mathrm{Lev}_\eta(T)$.
    - In the latter case a tower is added inside x, so nodes in T are subsets of x
    - The former case yields a contradiction, as then $T \restriction x$ is essentially a partition-type tree of maximal antichains in $[x]^\kappa$. But for $\alpha \in x$ there is a unique nodal element on every level containing it, resulting in a branch through $T \restriction x$ with nonempty intersection.

Proof Observations

- $\mathfrak{a}_\kappa = 2^\kappa$ is important
- Only $\subseteq$ was used because the tree is short; $\subseteq^*$ is not needed

Question 3: A tall base tree (height κ^+)

Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. If there is a tower of length κ^+, then there exists a base tree of height κ^+ for κ with levels of cardinality 2^κ.

- Essentially do the same thing as before, except add κ^+-length $\subseteq^*$-towers below elements in the root node families.
  - These tower sequences can no longer be continuous at limits and so there will be root node families at limit levels.
  - These "path-type" root node families are handled a bit differently than the "successor-type" families.
- To show maximality of the levels, the $\mathrm{cf}(\xi) = \kappa$ case has to be distinguished and the lemma that $T^\kappa_{<\omega} \subseteq {}^{<\kappa}\kappa$ contains no κ-Aronszajn subtree is needed.
- To show the base property of the tree, the lemma that a pruned tree T of height κ^+ with $|\mathrm{Lev}_\alpha(T)| < \kappa$ for every $\alpha < \kappa^+$ is eventually nonsplitting is needed.

Question 3: A base tree of height κ

Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. There exists a base tree of height κ for κ with levels of cardinality 2^κ.

- Essentially do the same thing as for the short base trees, except that instead of adding a single μ-length $\subseteq$-tower for relevant x below a suitable $r \in [\kappa]^\kappa$ in the root node families, we add κ-many $\subseteq$-towers for all relevant x.
  - The set of the lengths of these towers is cofinal in κ.
  - Unlike as in the short base trees construction, at intermediate limit lengths many tower paths are maximal (empty intersection) as these towers of varying length expire.
  - The no κ-Aronszajn subtrees of $T^\kappa_{<\omega} \subseteq {}^{<\kappa}\kappa$ lemma is required to show the base property.

Question 3: Base trees along the tower spectrum

Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. If $\lambda > \kappa$ is the regular length of a tower, there exists a base tree of height λ for κ with levels of cardinality 2^κ.

- If $\lambda > \kappa^+$, the new construction stage case to consider is where $\mathrm{cf}(\xi) > \kappa$. The handling requires again the lemma that pruned trees too tall relative to their width are eventually nonsplitting.

- The handling for $\mathrm{cf}(\xi) = \kappa$ and $\mathrm{cf}(\xi) < \kappa$ is the same as previous cases.

Note: One can add $\subseteq^*$-towers of varying lengths as in the construction of the tree of height κ to build a base tree of height $\lambda > \kappa$ the limit of (regular) lengths of towers too.

Unaddressed questions

- Are there methods for building non-partition type distributivity trees under weaker assumptions than $\mathfrak{a}_\kappa = 2^\kappa$?
  - This assumption is tied to the base property in these constructions.
- Non-existence consistencies for distributivity trees?
  - Heights $<\kappa$, κ, and $>\kappa$ each of interest.

Thank you CUNY