More Borel chromatic numbers

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October 25, 2024

Definable graphs

Definition

A *graph* is a set V of vertices together with a set $E \subseteq [V]^2$ of edges.

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the *edge-relation* $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

We focus on the lowest interesting complexity classes: clopen graphs, closed graphs, and F_σ-graphs.

Cardinal invariants

Definition

Let $G = (X, E)$ be a graph. Then $A \subseteq X$ is a *G-clique* (a *clique* in G) if $[A]^2 \subseteq E$.

$A \subseteq X$ is *G-independent* (an *independent* set in G) if $[A]^2 \cap E = \emptyset$. (Independent sets are sometimes called discrete.)

$A \subseteq X$ is *G-homogeneous* (a *homogeneous* set in G) if A is either independent or a clique in G.

Definition

The *clique-number* of a graph G is the supremum of the sizes of all G-cliques.

Clique-numbers are degenerate for graphs of low complexity:

Theorem (Kubiś)

A G_δ-graph with an uncountable clique has a perfect clique.

This is sharp: there is an F_σ-graph on 2^ω with a clique of size $\aleph_1$ but no perfect clique. The graph is a variant of the symmetrization of Turing reducibility (Folklore, Kubiś-Shelah, Mátrai).

Definition

The *chromatic number* of a graph G is the least size of a family of G-independent (G-discrete) sets that covers all the vertices of G.

The chromatic number of open graphs is degenerate in the following sense: An open graph is either countably chromatic or has a perfect clique and hence chromatic number $2^{\aleph_0}$ (provable instance of Todorcevic's OCA).

This dichotomy fails for closed graphs.

Definition

The *cochromatic number* of a graph $G = (V, E)$ is the least cardinality of a family of homogeneous sets that covers V.

Theorem

a) There is a clopen graph G_{min} on 2^ω such that a clopen graph G on a Polish space has an uncountable cochromatic number iff G_{min} embeds into G as an induced subgraph (GKKS).
b) There is a clopen graph G_{max} on 2^ω whose cochromatic number is maximal among all cochromatic numbers of clopen graphs on Polish spaces (GGK).
c) It is consistent that the cochromatic number of G_{max} is $\aleph_1 < 2^{\aleph_0}$ (GKKS).
d) It is consistent that G_{min} and G_{max} have different cochromatic numbers (GGK).

Definition

For any graph G let $\mathrm{Age}(G)$ denote the class of finite graphs that embed into G.

Theorem

Let G be a clopen graph on a Polish space. If $\mathrm{Age}(G)$ is generated by a finite set of finite graphs by taking isomorphic copies, induced subgraphs, and substitution, then the cochromatic number of G is countable or equal to the cochromatic number of G_{min}.

Example

$\mathrm{Age}(G_{\mathrm{min}})$ is generated by two graphs with two vertices: the edge and the non-edge.
$\mathrm{Age}(G_{\mathrm{max}})$ consists of all finite graphs.

A natural open question is whether there are any clopen graphs with an uncountable cochromatic number that is consistently different from those of G_{min} and G_{max}.

One strategy to solve this would be to find a class of finite graphs that is closed under isomorphisms, induced subgraphs, and substitution that is sufficiently different from the class of all finite graphs and every class generated by finitely many graphs.

A candidate is the class of all perfect (finite) graphs. Here a finite graph is perfect if for each induced subgraph the chromatic number is equal to the clique number.

We would need to prove a Ramsey theorem between the class of perfect graphs and the class of P_4-free graphs.

Borel chromatic numbers

Definition

Let G be a graph on a Polish space X. The *Borel chromatic number* $\chi_B(G)$ is the smallest size of a family of G-independent Borel sets that covers X.

Theorem (G_0-Dichotomy by Kechris, Solecki, Todorcevic)

There is a closed graph G_0 on the Cantor space 2^ω such that for every analytic graph G on a Polish space exactly one of the following holds:

- $\chi_B(G) \leq \aleph_0$
- *There is a continuous homomorphism h from G_0 to G.*

(A graph homomorphism maps edges to edges, but may collapse non-edges.)

Definition

Choose sequences $s_n \in 2^n$ such that each finite sequence $t \in 2^{<\omega}$ is an initial segment of some s_n. G_0 is the graph on 2^ω whose edges are of the form $\{s_n{}^\frown 0^\frown x, s_n{}^\frown 1^\frown x\}$ for some $x \in 2^\omega$.

G_0 is a forest, i.e., it does not have any cycles.

Hence the chromatic number of G_0 is 2.

The measurable chromatic number of G_0 is 3 (B. Miller).

What is the Borel chromatic number of G_0?

A lower bound is $\mathrm{cov}(\mathcal{M})$, the least size of a family of meager sets that covers 2^ω.

Note that G_0 does not have a perfect clique.

Theorem

Let G be a closed graph on a Polish space X.
Then either G has a perfect clique or there is a ccc forcing extension of the set-theoretic universe where $\chi_B(G) = \aleph_1$ while $2^{\aleph_0}$ is arbitrarily large.

How can this be done?
We first observe that it is enough to show this for $X = \omega^\omega$.
Now use a countable product with finite supports to add countably many closed independent ground model sets covering the ground model ω^ω.
This forcing turns out to be ccc. Iterate to get the desired model of set theory.

The last theorem in particular shows that $\chi_B(G_0)$ is consistently less than $2^{\aleph_0}$.
So, a natural question is whether there is another closed graph without perfect cliques whose Borel chromatic number is consistently different from that of G_0.

Definition

Let G_1 be the graph on 2^ω whose edges are of the form $\{x, y\}$ where $x, y \in 2^\omega$ differ at exactly one $n \in \omega$.

Note that G_1 is something like a homogeneous version of G_0.

Also note that this suggests an obvious definition of graphs G_n for all $n > 0$.

Every G_1-independent set is G_0-independent.

In particular, $\chi_B(G_0) \leq \chi_B(G_1)$, which also follows from the G_0-dichotomy.

Like G_0, the graph G_1 does not have any perfect cliques.

It turns out that the Borel chromatic number of G_1 is closely connected to Silver forcing.

Definition

Silver forcing $\mathbb{V}$ consists of partial functions from a coinfinite subset of ω to 2, ordered by reverse inclusion.
Every function p from a coinfinite subset of ω to 2 defines a *Silver set* $[p]$ consisting of all $x \in 2^\omega$ that extend p.
A *Silver tree* is a subtree T of $2^{<\omega}$ consisting of the initial segments of the elements of a Silver set.
Let I_{G_1} denote the σ-ideal on 2^ω generated by G_1-independent Borel sets.

Theorem (Zapletal)

An analytic set $A \subseteq 2^\omega$ is not in I_{G_1} iff A contains the branches of a Silver tree.

Zapletal's theorem shows that Silver forcing $\mathbb{V}$ is the optimal forcing to increase the Borel chromatic number of G_1.

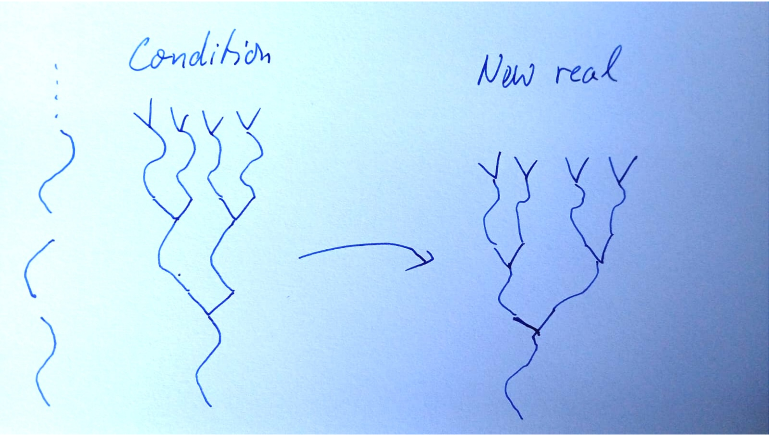
Theorem (GG, Zapletal)

Forcing with a countable support iteration of Silver forcing of length ω_2 yields a model of

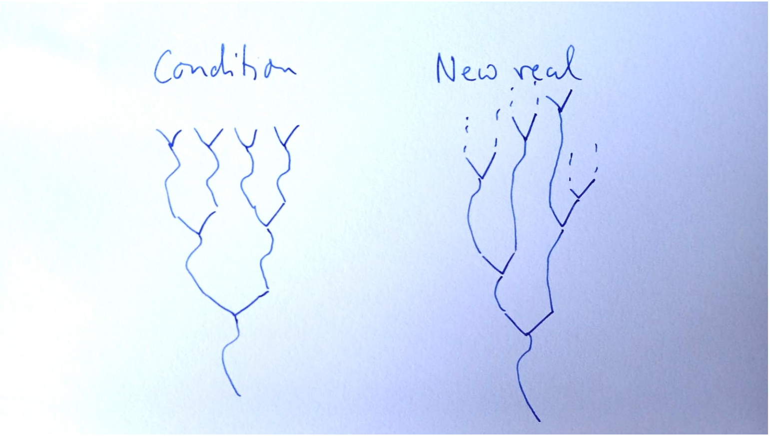
$$\aleph_1 = \chi_B(G_0) < \chi_B(G_1) = \aleph_2 = 2^{\aleph_0}.$$

The crucial point here is that every new real added by Silver forcing is contained in a closed G_0-independent set from the ground model.

First case: The new real looks like the Silver generic



Second case: The new real looks very different



I conjecture that all the graphs G_n, $n > 0$, have the same Borel chromatic number.
The natural forcing notions to separate those Borel chromatic numbers do not work as they are forcing equivalent to Silver forcing which increases $\chi_B(G_1)$.

There is a natural forcing notion to increase $\chi_B(G_0)$, but that is not proper.

However, there is modified version of this forcing notion that increases $\chi_B(G_0)$ and is proper.

Further consistency results:

Theorem (GG)

Consistently $\mathfrak{d} < \chi_B(G_0)$.

Theorem (Banerjee, Gaspar)

Iterated Laver forcing does not increase $\chi_B(G_1)$.
In particular, adding a Laver real does not add Silver reals.

The Turing graph

Definition

We identify subsets of ω with elements of 2^ω..
For $x, y \in 2^\omega$ and an oracle Turing machine M we write $x \leq_M y$ if the machine M, equipped with the oracle y, decides the language x.
An oracle Turing machine M is total if for all $y \in 2^\omega$ there is x such that $x \leq_M y$.
The *Turing graph* is the graph G_T on 2^ω where two distinct vertices x and y form an edge iff for some oracle Turing machine M $x \leq_M y$ or $y \leq_M x$.
The *total Turing graph* is the graph G_T^{total} on 2^ω where two distinct vertices x and y form an edge iff for some total oracle Turing machine M $x \leq_M y$ or $y \leq_M x$.

Theorem

The Turing graph is $G_{\delta\sigma}$ and the total Turing graph is F_σ.

Theorem

G_T^{total} and G_T both have an uncountable clique, but no perfect clique.

Theorem

Neither G_T^{total} nor G_T are G_δ.

Theorem (Mátrai?)

G_T is not F_σ.

Theorem
The chromatic numbers of G_T^{total} and G_T are both $\aleph_1$.

Theorem
It is consistent that $2^{\aleph_0}$ is arbitrarily large and $\chi_B(G_T^{\text{total}}) < 2^{\aleph_0}$.

It remains open whether it is possible to separate the Borel chromatic numbers of the Turing graph and the total Turing graph.

It is also unclear whether $\chi_B(G_T) < 2^{\aleph_0}$ is consistent or whether $\chi_B(G_0)$ and $\chi_B(G_T)$ can be separated.

Thank you!