Explicit models of arithmetic do not have full standard system

February 2024

MOPA Seminar

Elliot Glazer

Introduction

It is well-known under ZFC that there is a nonstandard model of PA which has a full standard system, i.e. every subset of this model's standard cut is the intersection of the standard cut with some subset of the model which is definable from parameters. We show that the use of Choice here cannot be avoided, and that there is no Borel model of arithmetic with a full standard system, answering a question of Kanovei.

Borel models

A Borel model of a theory of arithmetic T consists of a Polish space P and Borel functions $+$ and $\cdot$ such that $(P, +, \cdot) \models T$.

A Borel$^-$ model of T is a tuple $(P, +, \cdot, \sim)$, where $+, \cdot$ are Borel functions, $\sim$ a Borel equivalence relation, and $((P/\sim), +, \cdot) \models T$.

We can similarly define projective and projective$^-$ models of T.

Main results

1. ($\Pi^1_1 - \mathrm{CA}_0$) No Borel$^-$ model of EFA has full standard system.
2. (Z_2 + Projective Measurability + measure 1 uniformization) No projective$^-$ model of EFA has full standard system.
3. It is relatively consistent with Z_2 that no projective$^-$ model of EFA has full standard system.
4. In the Solovay model, there is no model of EFA with full standard system.
5. It is relatively consistent with ZF + DC that there is no model of EFA with full standard system.

Here EFA is the system $Q + \mathrm{I}\Delta_0 + \mathrm{Exp}$.

Beginning the proof of (2).

Assume projective measurability and measure 1 uniformization.

Suppose towards contradiction M is a projective$^-$ model of EFA with full standard system. Let X be a measure 1 subset of 2^ω and $F : X \to M \setminus \omega$ a function assigning each $x \in X$ a code in M. Let $\mathrm{Ext}(x)$ be the nonstandard binary string extending x which is coded by $F(x)$.

Let $x \in X$ be chosen uniformly at random. All probabilities mentioned throughout this argument are with respect to this random choice.

Notation

For two finite binary strings s and t, and $n \leq \omega$, let $s^\frown{}^n t$ denote the concatenation of s followed by n copies of t. Let $a_n = (0)^\frown{}^n(1)$, i.e. 0 followed by n 1's. Note that these terms are definable inside M for nonstandard n as well. Let $s^\frown{}^\infty t$ be $(s^\frown{}^\omega t)^M$.

Overspill lemma

A Fubini argument shows the following:

Overspill lemma:
For finite strings s and t (t of positive length), and $\epsilon > 0$, there exists a nonstandard N such that, with probability greater than $1 - \epsilon$, $\mathrm{Ext}(x + (s^\frown{}^\omega t))$ agrees with $\mathrm{Ext}(x) + s^\frown{}^\infty t$ below N.

The model has countable downward cofinality

For $n < \omega$, let $u_n(x)$ be the least $N > n+1$ such that $\mathrm{Ext}(x)(N) \neq \mathrm{Ext}(x + a_n)(N)$ (or set $u_n(x) = \infty$ if there is no such N). Notice N is necessarily nonstandard.

Cofinality lemma: With probability 1, $\langle u_n(x) \rangle$ is downwards cofinal in $M \setminus \omega$.

Proof of cofinality lemma

For $n < \omega$, $p < 1$, consider the set

$$I_{n,p} = \{m \in M : \mathrm{Prob}(u_n(x) < m) > p\}.$$

Clearly this is a final segment of $M \setminus \omega$. Applying the overspill lemma, we see $I_{n,p} \subsetneq M \setminus \omega$. Let

$$I_p = \bigcup_{n<\omega} I_{n,p}.$$

Proof of cofinality lemma (cont.)

Now we will show $I_p = M \setminus \omega$. Suppose not. Applying the overspill lemma, there is $m \in M \setminus (I \cup \omega)$ such that with probability greater than $1 - \frac{1-p}{2}$,

$$\mathrm{Ext}(x + a_\omega)(m) = \mathrm{Ext}(x)(m) + a_m(m) \neq \mathrm{Ext}(x)(m).$$

Let n be such that

$$\mathrm{Prob}(\mathrm{Ext}(x + a_\omega)(m) = \mathrm{Ext}(x + a_n)(m)) > 1 - \frac{1-p}{2}.$$

Then $\mathrm{Prob}(\mathrm{Ext}(x + a_n)(m) \neq \mathrm{Ext}(x)(m)) > p$, which implies $u_n \leq m$, so $m \in I_{n,p}$, contradiction.

Proof of cofinality lemma (cont.)

Let i_n be least such that

$$I_{n,1/2} \subsetneq I_{i_n,1-3^{-i}}.$$

Then with probability at least $1 - 3^{-i}$, $u_{i_n}(x) \notin I_{n,1/2}$. By the Borel-Cantelli lemma, with probability 1, for cofinitely many n, $u_{i_n}(x) \notin I_{n,1/2}$. Since $I_{1/2} = M \setminus \omega$, this proves the Lemma.

Setting up the contradiction

Let $\langle N_i \rangle$ be a downwards cofinal ω-sequence in $M \setminus \omega$. We will recursively construct $0 < i_n, k_n < \omega$, and finite strings s_n, where s_0 is the empty string and $s_{n+1} = s_n{}^\frown{}^{k_n} a_n$. Suppose we have constructed k_m for $m < n$ (and in particular have constructed s_n). Per the overspill lemma, let i_n be least such that, with probability greater than $1 - 3^{-n}$, $\mathrm{Ext}(x + (s_n{}^\frown{}^\omega a_n))$ agrees with $\mathrm{Ext}(x) + (s_n{}^\frown{}^\infty a_n)$ below $2i_n$.

Defining sequences

Let $k = k_n$ and $f : 2^k \to 2^{2n}$ be such that, with probability greater than $1 - 3^{-n}$, for all $j < 2n$, $\mathrm{Ext}(x)(N_{i_n} + j) = f(x \upharpoonright k)(j)$.

Let $s_\omega = \bigcup_{n<\omega} s_n$. We will show that with probability 1, we can define the standard cut from $\mathrm{Ext}(x)$ and $\mathrm{Ext}(x + s_\omega)$, which will be a contradiction.

A calculation

Fix n. With probability greater than $1 - 3 \cdot 3^{-n}$, for all $j < 2n$, we have

$$\begin{aligned}
\mathrm{Ext}(x + s_\omega)(N_{i_n} + j) &= f((x + s_\omega) \restriction k_n)(j) = f((x + s_{n+1}) \restriction k_n)(j) \\
&= f((x + (s_n {}^\frown{}^\omega a_n)) \restriction k_n)(j) \\
&= \mathrm{Ext}(x + (s_n {}^\frown{}^\omega a_n)(N_{i_n} + j)) \\
&= \mathrm{Ext}(x)(N_{i_n} + j) + (s_n {}^\frown{}^\infty a_n)(N_{i_n} + j).
\end{aligned}$$

(The first, fourth, and fifth equalities each have have probability greater than $1 - 3^{-n}$ of holding for all $j < 2n$.)

Agreement of x and $x + s_\omega$

By Borel-Cantelli, with probability 1, there are cofinitely many n such that

$$\mathrm{Ext}(x + s_\omega)(N_{i_n} + j) = \mathrm{Ext}(x)(N_{i_n} + j) + (s_n{}^\frown{}^\infty a_n)(N_{i_n} + j).$$

There are two $j < 2n$ such that $(s_n{}^\frown{}^\infty a_n)(N_{i_n} + j) = 0$. For these j, we have

$$\mathrm{Ext}(x)(N_{i_n} + j) = \mathrm{Ext}(x + s_\omega)(N_{i_n} + j).$$

The contradiction

In M we define the sequence $\langle b_n \rangle$ by letting b_n be the nth element of the set

$$B := \{b \in M : \mathrm{Ext}(x)(b) = \mathrm{Ext}(x + s_\omega)(b)\}.$$

With probability 1, there are cofinally many n such that

$$|B \cap \{N_{i_n} + j : j < 2n\}| = 2.$$

Thus,

$$\omega = \{n \in M : \forall k < n(b_{k+1} - b_k \leq b_{k+2} - b_{k+1})\}.$$

We have defined the standard cut, contradiction!

Relative consistency with ZF

Solovay's model requires an inaccessible cardinal to be constructed. Now we get rid of the large cardinal hypothesis.

Working in L, let $\mathbb{P}$ be the forcing which adds ω_2 many random reals, and let G be a generic filter. Let

$$V = L(\mathcal{P}(\omega_1))^{L[G]}.$$

Then $V \models \mathsf{ZF} + \mathsf{DC}_{\omega_1} + \mathsf{Unif}$. Furthermore, in this universe, there is a total extension μ of Lebesgue measure which is translation-invariant, has the Fubini property, and the Lebesgue density property. This is sufficient to formalize the probabilistic argument that there is no model of arithmetic which has full standard system.

Resolving Kanovei's problem

Kanovei asked whether there is a Borel model of arithmetic with full standard system. Our argument can be adjusted to prove in ZF (in fact, using just a fragment of second-order arithmetic) that there is no such model.

Suppose towards contradiction that c is a Borel code for a model of EFA with full standard system. This is a Π^1_2 assertion about c. By Shoenfield absoluteness, c codes a model of EFA with full standard system in $L[c]$.

A forcing extension

We do not have enough measure theory in this model to directly formalize our previous argument. Instead, consider the random real forcing extension $L[c][r]$. Applying absoluteness again, it holds in this model that c is a Borel code for a model M of EFA with full standard system.

Use coanalytic uniformization to define (with c but not r as a parameter) a choice of codes $F : \mathcal{P}(\omega) \to M \setminus \omega$, and as before let $\mathrm{Ext}(x)$ be the nonstandard binary string extending x which is coded by $F(x)$.

Then each probabilistic assertion about r can be formalized in $L[c]$ as the Lebesgue measure of the Boolean truth value of that assertion. In particular, we have probability 1 that in $L[c][r]$, M is a model of arithmetic which defines its standard cut, contradiction!

Implications for Scott's problem

Recall Scott's result that the countable Scott sets are precisely the countable subsets which are the standard system of some nonstandard model of PA. Scott asked whether this characterization holds with the countability hypothesis dropped.

Knight and Nadel confirmed that ZFC + CH resolves Scott's problem positively, and it remains open whether this is a ZFC theorem.

Trivially, $\mathcal{P}(\omega)$ is a Scott set, so Scott's problem cannot be decided in ZF. In light of the model we used for this separation, we ask

Question: Does the theory ZFC + "$\mathbb{R}$ is a real-valued measurable cardinal" negatively resolve Scott's problem?

Reference

Ali Enayat's 2009 notes "Borel structures via models of arithmetic (and set theory)."

Thank you for listening to my talk!