The ω-iterated nonstandard extension of $\mathbb{N}$ and Ramsey combinatorics

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Content

Three universes

Classically, all mathematical objects (sets) are elements of the universal class V of all sets. The scheme of the universe is very simple:

$$\mathsf{V}$$

In the nonstandard perspective the picture becomes little bit more complicated:

$$\begin{tikzcd} \mathsf{S} & \underset{\textit{elementary}}{\prec} & \mathsf{I} & \underset{\textit{transitive}}{\subseteq_t} & \mathsf{V} \arrow[llll, bend right, "*"', "\textit{isomorphism}"] \end{tikzcd}$$

where I is *almost universal* and κ*-saturated* for some uncountable cardinal κ.

Three universes

$$\begin{tikzcd} \mathsf{S} \arrow[r, phantom, "\prec"', "\textit{elementary}" below=6pt] & \mathsf{I} \arrow[r, phantom, "\subseteq_t", "\textit{transitive}" below=6pt] & \mathsf{V} \arrow[ll, bend right, "{*}", "\textit{isomorphism}"'] \end{tikzcd}$$

I is κ-saturated and almost universal

The main idea here is that we would like to **extend** V **by adding new ideal elements** (such as infinitesimals in $\mathbb{R}$). Instead of extending V directly, which is impossible, we **create an isomorphic copy** S **of** V inside of itself and **extend the copy**, creating a larger universe I, that contains all the ideal elements we need.

Thus S consists of elements ${}^*x$ for $x \in \mathsf{V}$ that are called standard and S the standard universe. The so called internal universe I contains all the nonstandard elements of sets ${}^*x \in \mathsf{S}$. The original V is also called the external universe.

Note: I can be constructed as an ultrapower $\mathsf{V}^A/\mathcal{U}$ with a carefully chosen ultrafilter $\mathcal{U}$ on an index set A. Then S consists of the equivalence classes $[c_x]_{\mathcal{U}}$ of constant functions $c_x : A \to \mathsf{V}$ with $x \in \mathsf{V}$, where $c_x(a) = x$ for all $a \in A$ and each $x \in \mathsf{V}$. The isomorphism ${}^* : \mathsf{V} \to \mathsf{S}$ is defined by ${}^*x = [c_x]_{\mathcal{U}}$ for $x \in \mathsf{V}$.

Three universes

$$\begin{tikzcd} \mathsf{S} \arrow[r, phantom, "\prec", "\textit{elementary}"'] & \mathsf{I} \arrow[r, phantom, "\subseteq_t", "\textit{transitive}"'] & \mathsf{V} \arrow[ll, bend right, "*", "\textit{isomorphism}"'] \end{tikzcd}$$

I is ⓚ-saturated and almost universal

By $\in$-formula we further mean a formula in the language of set theory. For φ an $\in$-formula φ^{X} is the relativization of φ to X obtained by replacing all quantifiers $\forall x, \exists x$ in φ by $\forall x \in \mathsf{X}, \exists x \in \mathsf{X}$.

Definition

$^* : \mathsf{V} \to \mathsf{S}$ isomorphism	$\Leftrightarrow$	$\varphi^{\mathsf{V}}(\bar{a}) \leftrightarrow \varphi^{\mathsf{S}}(^*\bar{a})$ for $\in$-formula φ, $\bar{a} \in \mathsf{V}$, * is injective and onto $\mathsf{S} = \mathrm{rng}(^*) = {}^*[\mathsf{V}]$,
$\mathsf{S} \prec \mathsf{I}$ (elementary)	$\Leftrightarrow$	$\varphi^{\mathsf{S}}(\bar{a}) \leftrightarrow \varphi^{\mathsf{I}}(\bar{a})$ for $\in$-formula φ and $\bar{a} \in \mathsf{S}$,
$\mathsf{I} \subseteq_t \mathsf{V}$ (transitive)	$\Leftrightarrow$	$y \in x \in \mathsf{I} \to y \in \mathsf{I}$,
I almost universal	$\Leftrightarrow$	$x \subseteq \mathsf{I} \to (\exists y \in \mathsf{I})(x \subseteq y)$,
I ⓚ-saturated	$\Leftrightarrow$	$(\mathcal{C} \subseteq \mathsf{I}$ centered & $\lvert\mathcal{C}\rvert < ⓚ) \to \bigcap \mathcal{C} \neq \emptyset$ where a system $\mathcal{C}$ is centered if for any finite $\mathcal{C}' \subseteq \mathcal{C}$ it is $\bigcap \mathcal{C}' \neq \emptyset$

Fundamental principles of nonstandard methods

Proposition (The transfer principle for Δ_1-formulas; $\mathrm{TP}(\Delta_1)$)

Let $\varphi(\overline{x})$ be a Δ_1-formula, then $(\forall \overline{x})(\varphi(\overline{x}) \leftrightarrow \varphi({}^*\overline{x}))$.

Proposition (*-absoluteness of Δ_1-definable relations and operations)

The following holds:

1. *Let R be a relation defined by a Δ_1-formula φ, i.e. let $R(\overline{x}) \leftrightarrow \varphi(\overline{x})$ holds for all $\overline{x}$. Then $R(\overline{x}) \leftrightarrow R({}^*\overline{x})$.*
2. *Let o be an operation defined by a Δ_1-formula φ, i.e. let $y = o(\overline{x}) \leftrightarrow \varphi(y, \overline{x})$ holds for all $y, \overline{x}$. Then ${}^*o(\overline{x}) = o({}^*\overline{x})$.*

Corollary (*-absoluteness of Δ_1-comprehension)

Let $u, \overline{z}$ be arbitrary sets and φ be a Δ_1-formula. Then

$$^*\{x \in u; \varphi(x, \overline{z})\} = \{x \in {}^*u; \varphi(x, {}^*\overline{z})\}.$$

Fundamental principles of nonstandard methods

Proposition

${}^\sigma x = {}^*x \cap \mathsf{S} = \{{}^*u; u \in x\}$.

Proposition (The standardization principle)

For any $z \subseteq \mathsf{S}$ there is a unique $y \in \mathsf{S}$ such that $y \cap \mathsf{S} = z$.

Proposition

The following dichotomy holds:

1. *If x is finite, then ${}^*x = \{{}^*u; u \in x\}$ and thus ${}^*x \subseteq \mathsf{S}$.*
2. *If x is infinite, then ${}^*x \supsetneq \{{}^*u; u \in x\}$ and thus ${}^*x$ contains a nonstandard element $v \notin \mathsf{S}$. Moreover there are at least $\textcircled{\kappa}$ nonstandard elements in ${}^*x$.*

Iterating the mapping *

In what follows, we are going to work not only with the first nonstandard extensions ${}^*x$ of sets x but with higher extensions ${}^{**}x$, ${}^{***}x$, ... as well.

We therefore define the (class) mappings ${}^{n*} : \mathrm{V} \to \mathrm{V}$, for all $n \in \mathbb{N}$, by induction

$$ {}^{0*}x = x, \qquad {}^{(n+1)*}x = {}^*({}^{n*}x) $$

and the (class) mapping ${}^\bullet : \mathrm{V} \to \mathrm{V}$ by

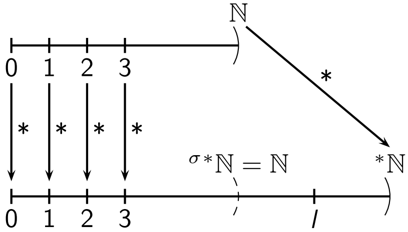
$$ {}^\bullet x = \bigcup_{n \in \mathbb{N}} {}^{n*}x. $$

In our notation, the operations ^{n*} and ${}^\bullet$ have the highest priority. For instance, ${}^\bullet\mathbb{N}^{<\omega}$ means $({}^\bullet\mathbb{N})^{<\omega}$ and similarly for ^{n*}.

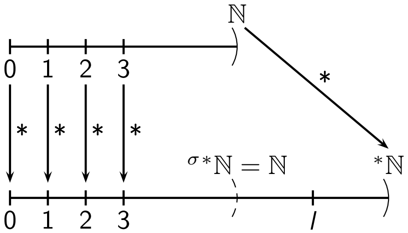
Note: We have defined the mappings ^{n*} not just for meta-mathematical n but really for all $n \in \mathbb{N}$. Compare this with defining ${}^{n*}x$ simply as ${}^{**\cdots*}x$ where * appears n-times, which would define ^{n*} only for meta-mathematical n. (Of course for meta-mathematical n these two definitions are equivalent.)

Natural numbers in the nonstandard perspective

The behavior of the mapping * and the mutual relation of the sets $\mathbb{N}$ and ${}^*\mathbb{N}$ can be summarized by the following picture:



Natural numbers in the nonstandard perspective

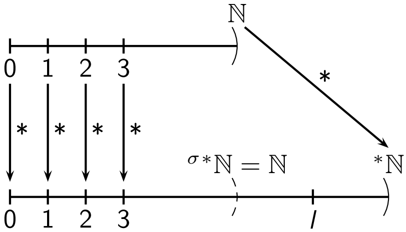


Proposition

The mapping * is the identity on the set $\mathbb{N}$, i.e. $^*n = n$ for any $n \in \mathbb{N}$.

Proof: By induction on $\mathbb{N}$. For $n = 0$, we have $^*0 = {^*\emptyset} = \emptyset = 0$, and for the induction step, $^*(n+1) = {^*(n \cup \{n\})} = {^*n} \cup \{^*n\} = n \cup \{n\} = n + 1$ by absoluteness of Δ_1-definable operations.

Natural numbers in the nonstandard perspective

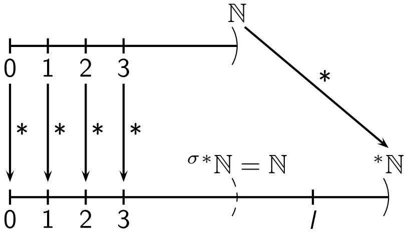


Corollary

The following are true:

1. $\mathbb{N} \subseteq {}^*\mathbb{N}$,
2. $^*\mathbb{N} \cap \mathsf{S} = \mathbb{N}$,
3. $|{}^*\mathbb{N} - \mathbb{N}| \geq \textcircled{\kappa}$.

Natural numbers in the nonstandard perspective



Proposition

The following holds true:

1. *a)* $n \in \mathsf{S}$ *for any* $n \in \mathbb{N}$, *b)* $I \in \mathsf{I} - \mathsf{S}$ *for any* $I \in {}^*\mathbb{N} - \mathbb{N}$,
2. *a)* ${}^*\mathbb{N} \in \mathsf{S}$, *b)* ${}^*\mathbb{N} - \mathbb{N} \notin \mathsf{I}$, *c)* $\mathbb{N} \notin \mathsf{I}$.

Natural numbers in the nonstandard perspective

Let us consider the structure $\underline{\mathbb{N}} = \langle \mathbb{N}, 0, S, +, \cdot, \leq \rangle$. We get ${}^*\underline{\mathbb{N}} = \langle {}^*\mathbb{N}, {}^*0, {}^*S, {}^*+, {}^*\cdot, {}^*\leq \rangle$ and, by the transfer principle ${}^*\underline{\mathbb{N}}$ is ⓚ-saturated elementary extension of $\underline{\mathbb{N}}$, i.e.

$$\underline{\mathbb{N}} \prec {}^*\underline{\mathbb{N}}.$$

A similar statement holds true for structures in the „full“ language of $\mathbb{N}$, that is the language that contains a symbol for every function and relation (of any arity) on $\mathbb{N}$.

Note: Arbitrary structure $\mathcal{A}$ in a finite language can be elementarily embedded into the ⓚ-saturated structure ${}^*\mathcal{A}$ in a similar way. Hence * gives an universal way for construction of ⓚ-saturated elementary extensions of structures.

ω-iterated nonstandard extension of $\mathbb{N}$

From $\mathbb{N} \subsetneq {}^*\mathbb{N}$, we get (by the transfer principle) the following chain of nonstandard extensions of $\mathbb{N}$:

$$\mathbb{N} \subsetneq {}^*\mathbb{N} \subsetneq {}^{**}\mathbb{N} \subsetneq \cdots \subsetneq {}^{\bullet}\mathbb{N}.$$

In fact, all these are end-extensions with respect to the usual ordering $\in$ of natural numbers.

Further on, letters α, β, ν usually denote elements of $^{\bullet}\mathbb{N}$, $\overline{\alpha}, \overline{\beta}, \overline{\nu}$ tuples of elements from $^{\bullet}\mathbb{N}$.

The rank function on $^{\bullet}\mathbb{N}$ is defined as

$$r(\alpha) = \min\{n \in \mathbb{N}; \alpha \in {}^{n*}\mathbb{N}\}, r(\overline{\alpha}) = \max\{r(\alpha_i); i < l(\overline{\alpha})\}.$$

Note: The rank function can be defined even more generally on the set $^{\bullet}V_\omega$ (where V_ω is the set of all hereditarily finite sets).

ω-iterated nonstandard extension of $\mathbb{N}$

Lemma (Transfer principle for ${}^{\bullet}\mathbb{N}$)

Let φ be a Δ_1-formula, $\overline{\alpha} \in {}^{\bullet}\mathbb{N}$, $m, n \geq r(\overline{\alpha})$. Then

$$\varphi(\overline{\alpha}, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{n*}\overline{y}).$$

Proof: By $n - m$ iterations of the transfer principle, we get

$$(\varphi(\bar{i}, \overline{y}) \leftrightarrow \varphi(\bar{i}, {}^{(n-m)*}\overline{y}))$$

for all $\bar{i} \in \mathbb{N}$.

By applying the transfer principle m times on the Δ_1 formula

$$(\forall \bar{i} \in \mathbb{N})(\varphi(\bar{i}, \overline{y}) \leftrightarrow \varphi(\bar{i}, {}^{(n-m)*}\overline{y}))$$

we get

$$(\forall \overline{\alpha} \in {}^{m*}\mathbb{N})(\varphi(\overline{\alpha}, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{n*}\overline{y})),$$

and we are done because $\overline{\alpha} \in {}^{r(\overline{\alpha})*}\mathbb{N} \subseteq {}^{m*}\mathbb{N}$.

ω-iterated nonstandard extension of $\mathbb{N}$

Corollary

Let $\mathcal{S}$ denote the set of all functions $g : \mathbb{N}^m \to \mathbb{N}$ and relations $R \subseteq \mathbb{N}^m$ of all arities m. The structures $\langle {}^{n*}\mathbb{N}, {}^{n*}s\rangle_{s\in\mathcal{S}}$ form the elementary chain

$$\langle \mathbb{N}, s\rangle_{s\in\mathcal{S}} \prec \langle {}^{*}\mathbb{N}, {}^{*}s\rangle_{s\in\mathcal{S}} \prec \langle {}^{**}\mathbb{N}, {}^{**}s\rangle_{s\in\mathcal{S}} \ldots \prec \langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}}$$

with the limit $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}}$.

Proof: Observe that for any $\mathcal{S}$-formula ψ, the formula

$$\langle {}^{n*}\mathbb{N}, {}^{n*}s\rangle_{s\in\mathcal{S}} \models \psi(\overline{\alpha})$$

is equivalent to a Δ_1 $\in$-formula

$$\varphi(\overline{\alpha}, {}^{n*}\mathbb{N}, {}^{n*}S),$$

where $S \subseteq \mathcal{S}$ is the finite set of symbols used in ψ. Then the transfer principle for ${}^{\bullet}\mathbb{N}$ can be applied.

Indiscernibility of *-chains

We say that an increasing sequence $(\beta_i; i \in \omega)$ of elements from ${}^*\mathbb{N}$ is indiscernible with respect to the formula $\varphi(\overline{x})$ if

$$\varphi(\beta_{i_0}, \dots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\beta_{j_0}, \dots, \beta_{j_{l(\overline{x})-1}})$$

whenever $i_0 < \dots < i_{l(\overline{x})-1}$, $j_0 < \dots < j_{l(\overline{x})-1}$.

We also say that $(\beta_i; i \in \omega)$ is strongly indiscernible with respect to the formula $\varphi(\overline{z}, \overline{x})$ if

$$(\forall \overline{z} \leq \beta_k)(\varphi(\overline{z}, \beta_{i_0}, \dots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\overline{z}, \beta_{j_0}, \dots, \beta_{j_{l(\overline{x})-1}}))$$

whenever $k < i_0 < \dots < i_{l(\overline{x})-1}$, $k < j_0 < \dots < j_{l(\overline{x})-1}$.

Indiscernibility of *-chains

The sequence $(\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \dots)$ with $\beta \in {}^\bullet\mathbb{N}$ satisfies the following indiscernibility property:

Proposition

Let $\beta \in {}^\bullet\mathbb{N}$ and φ be a Δ_1-formula. Then

$$\varphi(\overline{\alpha}, {}^{i_0 r(\beta)*}\beta, \dots, {}^{i_{k-1} r(\beta)*}\beta, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{i'_0 r(\beta)*}\beta, \dots, {}^{i'_{k-1} r(\beta)*}\beta, {}^{m'*}\overline{y})$$

for all $i_0 < \cdots < i_{k-1}$, $i'_0 < \cdots < i'_{k-1}$, all $\overline{\alpha}$ satisfying $r(\overline{\alpha}) \leq i_0 r(\beta), i'_0 r(\beta)$ and all m, m' such that $(i_{k-1}+1)r(\beta) \leq m$, $(i'_{k-1}+1)r(\beta) \leq m'$.

Proof: Technical, but just by application of the transfer principle for ${}^\bullet\mathbb{N}$.

Indiscernibility of *-chains

We say that $(\beta_i; i \in \omega)$ is rank-indiscernible with respect to $\varphi(\overline{z}, \overline{x})$ if

$$(\forall \overline{z})(r(\overline{z}) \leq r(\beta_k) \rightarrow (\varphi(\overline{z}, \beta_{i_0}, \ldots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\overline{z}, \beta_{j_0}, \ldots, \beta_{j_{l(\overline{x})-1}})))$$

whenever $k < i_0 < \ldots < i_{l(\overline{x})-1}$, $k < j_0 < \ldots < j_{l(\overline{x})-1}$.

Clearly any rank-indiscernible sequence is strongly indiscernible.

We say that a sequence is [strogly, rank-] indiscernible with respect to a formula $\varphi(\overline{x})$ of the structure $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s \rangle_{s \in \mathcal{S}}$ (i.e. $\varphi(\overline{x})$ is an $\mathcal{S}$-formula) if it is [strongly, rank-] indiscernible with respect to the $\in$-formula $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s \rangle_{s \in \mathcal{S}} \models \varphi(\overline{x})$.

Corollary

For every $\beta \in {}^{\bullet}\mathbb{N}$, the sequence $(\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \ldots)$ is rank-indiscernible with respect to all

1. *parameter-free Δ_1-formulas,*
2. *parameter-free formulas of $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s \rangle_{s \in \mathcal{S}}$.*

Indiscernibility of *-chains

Note: The previous result allows easy proofs of various Ramsey-type theorems in ${}^{\bullet}\mathbb{N}$. Take the Ramsey theorem for n-tuples as an example:

Proposition (Ramsey theorem on ${}^{\bullet}\mathbb{N}$)

Let $\mathcal{C}$ be a finite coloring of $\langle\mathbb{N}\rangle^n$. Then there is $C \in \mathcal{C}$ and an infinite set $A \subseteq {}^{\bullet}\mathbb{N}$ such that $\langle A\rangle^n \subseteq {}^{\bullet}C$.

Proof: Take any $\beta \in {}^{\bullet}\mathbb{N} - \mathbb{N}$ and set $A = \{\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \ldots\}$. Clearly $(\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \ldots, {}^{nr(\beta)*}\beta) \in {}^{\bullet}C$ for some $C \in \mathcal{C}$. Then the statement of the theorem follows by indiscernibility w.r.t. the $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}}$-formula $\overline{x} \in {}^{\bullet}C$.

Unfortunately Ramsey-type statements postulate existence of infinite sets and as such are not easily transferable from ${}^{\bullet}\mathbb{N}$ down to $\mathbb{N}$. We need to understand the structure of ${}^{\bullet}\mathbb{N}$ much better to see that this transfer is indeed possible.

Indistinguishability equivalence on ${}^\bullet\mathbb{N}$

An important equivalence on ${}^\bullet\mathbb{N}$ is that of indistinguishability by standard properties:

$$\alpha \sim \beta \leftrightarrow (\forall A \subseteq \mathbb{N})(\alpha \in {}^\bullet A \leftrightarrow \beta \in {}^\bullet A).$$

Note that $\sim$ is nontrivial (for example $\alpha \sim {}^{n*}\alpha$ for all n). In fact $\sim$ corresponds to the Čech-Stone compactification $\beta\mathbb{N}$ of $\mathbb{N}$.

We say that an equivalence $\approx$ on ${}^\bullet\mathbb{N}$ is a congruence with respect to a (partial) function $f : {}^\bullet\mathbb{N}^{<\omega} \nrightarrow {}^\bullet\mathbb{N}$ or a relation $P \subseteq {}^\bullet\mathbb{N}^{<\omega}$ if

$$\overline{\alpha} \approx \overline{\alpha}' \to f(\overline{\alpha}) \approx f(\overline{\alpha}') \quad \text{or} \quad \overline{\alpha} \approx \overline{\alpha}' \to (P(\overline{\alpha}) \leftrightarrow P(\overline{\alpha}'))$$

respectively, granted that, in the first case, at least one of $f(\overline{\alpha})$, $f(\overline{\alpha}')$ is defined.

Indistinguishability equivalence on ${}^{\bullet}\mathbb{N}$

It is almost an imperative for an equivalence relation, in order to be useful in algebraic calculations, **to be a congruence** with respect to the operations and relations that we want to compute with. Sadly, this is **not the case** of the indistinguishability relation $\sim$ on ${}^{\bullet}\mathbb{N}$ – the equivalence $\sim$ is not a congruence with respect to $+$ (nor w.r.t. multiplication nor w.r.t. most other functions of arity > 1):

Example: For $\nu \in {}^{\bullet}\mathbb{N}$ denote by $v_2(\nu)$ the largest μ such that $2^{\mu}|\nu$. Let $\alpha \in {}^*\mathbb{N} - \mathbb{N}$ be such that $\beta = v_2(\alpha) \notin \mathbb{N}$. Then $\alpha \sim {}^*\alpha$ but $\alpha + \alpha \not\sim \alpha + {}^*\alpha$. Let us prove the latter claim:
We have $v_2({}^*\alpha) = {}^*\beta > \beta$ and therefore $v_2(\alpha + {}^*\alpha) = \beta$. On the other hand $v_2(\alpha + \alpha) = v_2(2\alpha) = \beta + 1$. Define $A = \{x \in \mathbb{N}; v_2(x) \text{ is odd}\}$. Then we have $\alpha + \alpha \in {}^*A \leftrightarrow \alpha + {}^*\alpha \notin {}^{**}A$.

Luckily, there is a large class of functions and relations with respect to whom the equivalence $\sim$ is a congruence – the graded functions and relations on ${}^{\bullet}\mathbb{N}$.

Grading

We define the transformation (grading) $^{\uparrow} : {}^{\bullet}\mathbb{N}^{<\omega} \to {}^{\bullet}\mathbb{N}^{<\omega}$ by

$$\overline{\alpha}^{\uparrow} = (\alpha_0, {}^{r(\alpha_0)*}\alpha_1, {}^{(r(\alpha_0)+r(\alpha_1))*}\alpha_2, \ldots, {}^{(\sum_{i<l(\overline{\alpha})-1} r(\alpha_i))*}\alpha_{l(\overline{\alpha})-1}).$$

For every (partial) function $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ and every relation $R \subseteq \mathbb{N}^{<\omega}$ we set

$$g^{\uparrow}(\overline{\alpha}) = {}^{\bullet}g(\overline{\alpha}^{\uparrow}), \quad R^{\uparrow}(\overline{\alpha}) \leftrightarrow {}^{\bullet}R(\overline{\alpha}^{\uparrow}),$$

for all $\overline{\alpha} \in {}^{\bullet}\mathbb{N}$ such that the right side is defined.

As special, but very important cases, we get:

$$\alpha +^{\uparrow} \beta = \alpha + {}^{r(\alpha)*}\beta, \quad \alpha \cdot^{\uparrow} \beta = \alpha \cdot {}^{r(\alpha)*}\beta.$$

Indistinguishability congruence

Proposition

The equivalence $\sim$ is a congruence with respect to $s^{\uparrow}$ whenever s is a partial function $s : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ or a relation $s \subseteq \mathbb{N}^{<\omega}$.

Proof: Inductively, using the *-absoluteness of Δ_1-comprehension.

As for unary $g : \mathbb{N} \to \mathbb{N}$ we have by definition $g^{\uparrow} = {}^{\bullet}g$, we get the following trivial but important:

Corollary

Let $g : \mathbb{N} \to \mathbb{N}$ then $\sim$ is a congruence with respect to ${}^{\bullet}g$.

Topology of indistinguishability

We define

$$\widetilde{\mathbb{N}} = {}^\bullet\mathbb{N}/_\sim,\ \widetilde{g} = g^{\uparrow}/_\sim,\ \widetilde{R} = R^{\uparrow}/_\sim,\ \widetilde{\alpha} = [\alpha]_\sim$$

for $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$, $R \subseteq \mathbb{N}^{<\omega}$, $\alpha \in {}^\bullet\mathbb{N}$, and $\Upsilon : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\widetilde{\mathbb{N}})$ by

$$\Upsilon(A) = \{\widetilde{\alpha}; \alpha \in {}^\bullet A\}.$$

The set $\{\Upsilon(A); A \subseteq \mathbb{N}\}$ is a basis of a topology on $\widetilde{\mathbb{N}}$. We call this topology the canonical topology on $\widetilde{\mathbb{N}}$. The canonical topology is uniform with the basis of uniformity

$$\widetilde{\mathbb{W}} = \{\widetilde{w}_t; t \in [\mathcal{P}(\mathbb{N})]^{<\omega}\}\ ;\ \widetilde{w}_t = \{(\widetilde{\alpha}, \widetilde{\beta}); (\forall A \in t)(\widetilde{\alpha} \in \Upsilon(A) \leftrightarrow \widetilde{\beta} \in \Upsilon(A))\}.$$

Proposition ($\circledast\!\kappa > 2^\omega$)

1. *$(\widetilde{\mathbb{N}}, \widetilde{\mathbb{W}})$ is a uniform compact Hausdorff space.*
2. *All functions $\widetilde{g}$ with $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ are continuous in the first coordinate in $(\widetilde{\mathbb{N}}, \widetilde{\mathbb{W}})$.*

Topology of indistinguishability

Note: It is perhaps even more enlightening to define the same topological space using the theory of nonstandard topology as an indifference space on ${}^{*\bullet}\mathbb{N}$:

There are two natural extensions of $\sim$ from ${}^{\bullet}\mathbb{N}$ to ${}^{*\bullet}\mathbb{N}$ – the *-extension ${}^{*}\sim$ and the equivalence $\approx$ defined by

$$a \approx b \leftrightarrow (\forall A \subseteq \mathbb{N})(a \in {}^{*\bullet}A \leftrightarrow b \in {}^{*\bullet}A).$$

The topology of $(\widetilde{\mathbb{N}}, \widetilde{\mathbb{W}})$ is exactly the same as the topology corresponding to the indifference space

$$\langle {}^{*}\widetilde{\mathbb{N}}, \widetilde{\approx} \rangle = \langle {}^{*\bullet}\mathbb{N}, \approx \rangle /_{{}^{*}\sim} = \langle {}^{*\bullet}\mathbb{N}/_{{}^{*}\sim}, \approx /_{{}^{*}\sim} \rangle,$$

which is the canonical Hausdorffization of $\langle {}^{*\bullet}\mathbb{N}, \approx \rangle$. (It is easy to prove that ${}^{*}\sim \subsetneq \approx$ and that ${}^{*}\sim$ is the finest standard equivalence Q on ${}^{*\bullet}\mathbb{N}$ that makes the factor-space $\langle {}^{*\bullet}\mathbb{N}, \approx \rangle /_{Q}$ Hausdorff.)

Idempotent elements

The continuity of graded functions in the first coordinate is important as it guarantees (under additional assumptions) the existence of idempotent elements:

We say that $\alpha \in {}^{\bullet}\mathbb{N}$ is g-idempotent (with $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$) if $\widetilde{\alpha}$ is an idempotent element of $(\widetilde{\mathbb{N}}, \widetilde{g})$, i.e. if $\widetilde{g}(\widetilde{\alpha}, \ldots, \widetilde{\alpha}) = \widetilde{\alpha}$ whenever the left side is defined.

That is, α is g-idempotent iff $g^{\uparrow}(\alpha, \ldots, \alpha) \sim \alpha$.

In particular: A number α is additively or multiplicatively idempotent if

$$\alpha +^{\uparrow} \alpha = \alpha + {}^{r(\alpha)*}\alpha \sim \alpha \text{ or } \alpha \cdot^{\uparrow} \alpha = \alpha \cdot {}^{r(\alpha)*}\alpha \sim \alpha$$

respectively.

Existence of idempotents

A semigroup is a structure with a single binary associative operation.

Lemma (Ellis-Numakura)

Let S be a semigroup with a compact Hausdorff topology and such that the semigroup operation is left-continuous. Then S contains an idempotent element.
Moreover, every element of the minimal compact subsemigroup of S is idempotent.

Proposition ($\kappa > 2^{\omega}$)

Let $X \subseteq \mathbb{N}$ and $g : X^2 \to X$ associative. Then there is a g-idempotent element $\alpha \in {}^{\bullet}X$ (even $\alpha \in {}^{*}X$).

Proof: $\tilde{X} = {}^{\bullet}X/_{\sim}$ is a compact subset of $(\widetilde{\mathbb{N}}, \widetilde{\mathbb{W}})$ and thus $(\tilde{X}, \tilde{g})$ is a semigroup satisfying the conditions of the Ellis-Numakura Lemma.

Existence of idempotents

The following is a very important special case:

Corollary ($\textcircled{\kappa} > 2^{\omega}$)

There is an additively idempotent element $\alpha \neq 0$ and a multiplicatively idempotent element $\beta \neq 0, 1$ in ${}^{\bullet}\mathbb{N}$ (even in ${}^{*}\mathbb{N}$).

Direct proof of Hindmann's theorem

To illustrate the abstract general idea, we show a direct proof of Hindmann's theorem (another, even shorter, proof using the Witnessing principle will be shown later):

Theorem (Hindmann)

Let $\mathcal{C}$ be a finite coloring of $\mathbb{N}$. Then there are an infinite set $A \subseteq \mathbb{N}$ and $C \in \mathcal{C}$ such that $\sum x_i \in C$ for all $\emptyset \neq \overline{x} \in \langle A \rangle^{<\omega}$.

Proof: see the lecture notes, page 58

Direct proof of Hindmann's theorem

The crucial idea of the proof is that of using a nonstandard „witness" $\nu \in {}^*\mathbb{N} - \mathbb{N}$ for the existence of the infinite homogeneous set A.

In ${}^{\bullet}\mathbb{N}$, the indiscernible sequence $(\nu, {}^*\nu, {}^{**}\nu, \dots)$ already constitutes an infinite homogeneous set (for idempotent ν, by a similar argument we used earlier for the Ramsey's theorem). However in order to „collapse" this infinite set down to $\mathbb{N}$, we need to do some work.

The only assumption we need for this is the „witnessing property"

$$\nu \in C'^{\uparrow} \text{ and } (\overline{x}, \nu) \in C'^{\uparrow} \rightarrow (\overline{x}, \nu, \nu) \in C'^{\uparrow}$$

to be satisfied by ν. The property has an „inductive form" with a base and induction steps.

This idea of using a witnessing property of this kind to allow a collapse of a cofinal indiscernible sequence in ${}^{\bullet}\mathbb{N}$ onto an infinite homogeneous set in $\mathbb{N}$ can be abstracted out. The resulting statement then is the Witnessing principle.

The witnessing principles

We formulate two very similar variants of the Witnessing principle – one for the case where the length of tuples in question is bounded (such as in Ramsey's theorem), the other where it is unbounded (such as in Hindmann's or Milliken's and Taylor's theorems). Both of these are actually special cases of a single yet more abstract principle, that we do not formulate here. By $\overline{x}^{[n]}$ we denote the constant tuple $(x, x, \ldots, x)$ of length n.

Theorem (Witnessing principles)

1. *Let $K \subseteq \langle\mathbb{N}\rangle^n$ for some $n \in \mathbb{N} - \{0\}$. If there is a witness $\nu \in {}^{\bullet}\mathbb{N} - \mathbb{N}$ satisfying*
$$\overline{\nu}^{[n]} \in K^{\uparrow},$$
then there is an infinite $A \subseteq \mathbb{N}$ such that $\langle A\rangle^n \subseteq K$.

2. *Let $K \subseteq \langle\mathbb{N}\rangle^{<\omega}$, $n \in \mathbb{N} - \{0\}$. If there is a witness $\nu \in {}^{\bullet}\mathbb{N} - \mathbb{N}$ satisfying*
$$\overline{\nu}^{[n]} \in K^{\uparrow} \text{ and } (\forall \overline{m} \in \langle\mathbb{N}\rangle^{<\omega})((\overline{m}, \overline{\nu}^{[n]}) \in K^{\uparrow} \rightarrow (\overline{m}, \overline{\nu}^{[n+1]}) \in K^{\uparrow}),$$
then there is an infinite $A \subseteq \mathbb{N}$ such that $\langle A\rangle^{<\omega} \subseteq K$.

We show how easily many Ramsey type theorems follow from the Witnessing principles.

See the lecture notes, Section 9.4, from page 61

Thanks

Thank you for your attention.

More details in the lecture notes: https://glivicky.com/wp-content/uploads/2019/03/nonstandard_analysis.pdf