Σ_n-Correct Forcing Axioms

Ben Goodman
CUNY Graduate Center

March 31, 2023

Standard forcing axioms

Recall the standard method of obtaining a model of the forcing axiom for some suitable forcing class Γ from a model with a supercompact cardinal κ:

Standard forcing axioms

Recall the standard method of obtaining a model of the forcing axiom for some suitable forcing class Γ from a model with a supercompact cardinal κ:

Let $f : \kappa \to V_\kappa$ be a Laver function. We recursively construct a sequence of posets in Γ $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ and let $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ be the iteration of it with support suitable to Γ.

Standard forcing axioms

Recall the standard method of obtaining a model of the forcing axiom for some suitable forcing class Γ from a model with a supercompact cardinal κ:

Let $f : \kappa \to V_\kappa$ be a Laver function. We recursively construct a sequence of posets in Γ $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ and let $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ be the iteration of it with support suitable to Γ.

If $\mathbb{P}_\alpha$ has been defined for $\alpha < \kappa$, let $\mathbb{Q}_\alpha$ be $f(\alpha)$ whenever that is a $\mathbb{P}_\alpha$ name for a poset in Γ and trivial if $f(\alpha)$ is anything else.

Standard forcing axioms

Recall the standard method of obtaining a model of the forcing axiom for some suitable forcing class Γ from a model with a supercompact cardinal κ:

Let $f : \kappa \to V_\kappa$ be a Laver function. We recursively construct a sequence of posets in Γ $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ and let $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ be the iteration of it with support suitable to Γ.

If $\mathbb{P}_\alpha$ has been defined for $\alpha < \kappa$, let $\mathbb{Q}_\alpha$ be $f(\alpha)$ whenever that is a $\mathbb{P}_\alpha$ name for a poset in Γ and trivial if $f(\alpha)$ is anything else.

Let $G \subseteq \mathbb{P}_\kappa$ be a V-generic filter.

Provably persistent formulas

Provably persistent formulas

Definition

If Γ is a forcing class, a formula ϕ is said to be provably Γ-persistent if *ZFC* proves

$$\forall x(\phi(x) \rightarrow \forall \mathbb{Q} \in \Gamma \Vdash_{\mathbb{Q}} \phi(\check{x}))$$

V[G] contains Σ_2-correct filters

V[G] contains Σ_2-correct filters

Theorem

In $V[G]$, for every $\mathbb{P} \in \Gamma$, every collection $\mathcal{D}$ of fewer than κ dense subsets of $\mathbb{P}$, every $\mathbb{P}$-name $\dot{a} \in H_\kappa^{V[G]}$, and every provably Γ-persistent Σ_2 formula ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$ such that $V[G] \models \phi(\dot{a}^F)$.

V[G] contains Σ_2-correct filters

Theorem

In $V[G]$, for every $\mathbb{P} \in \Gamma$, every collection $\mathcal{D}$ of fewer than κ dense subsets of $\mathbb{P}$, every $\mathbb{P}$-name $\dot{a} \in H_\kappa^{V[G]}$, and every provably Γ-persistent Σ_2 formula ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$ such that $V[G] \models \phi(\dot{a}^F)$.

Proof.

We use the following fact:

Lemma

For any Σ_2 formula ϕ and any parameter b, $\phi(b)$ holds if and only if there is an uncountable cardinal θ such that $b \in H_\theta$ and $H_\theta \models \phi(b)$.

V[G] contains Σ_2-correct filters

Theorem

In $V[G]$, for every $\mathbb{P} \in \Gamma$, every collection $\mathcal{D}$ of fewer than κ dense subsets of $\mathbb{P}$, every $\mathbb{P}$-name $\dot{a} \in H_\kappa^{V[G]}$, and every provably Γ-persistent Σ_2 formula ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$ such that $V[G] \models \phi(\dot{a}^F)$.

Proof.

We use the following fact:

Lemma

For any Σ_2 formula ϕ and any parameter b, $\phi(b)$ holds if and only if there is an uncountable cardinal θ such that $b \in H_\theta$ and $H_\theta \models \phi(b)$.

Let $\dot{\mathbb{P}}$ and $\ddot{a}$ be appropriate $\mathbb{P}_\kappa$-names in V. By the lemma, there is a $\theta \gg |\dot{\mathbb{P}}|$ such that, for some $p \in G$,

$$H_\theta \models p \Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{P}} \in \Gamma \wedge (p, \dot{1}_{\mathbb{P}}) \Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{P}}} \phi(\ddot{a})$$

.

V[G] contains Σ_2-correct filters

V[G] contains Σ_2-correct filters

Proof.

(continued)

Now let $\lambda > |H_\theta|$ and fix an elementary embedding $j : V \to M$ such that:

- $crit(j) = \kappa$
- $j(\kappa) > \lambda$
- ${}^\lambda M \subset M$
- $j(f)(\kappa) = \dot{\mathbb{P}}$

V[G] contains Σ_2-correct filters

Proof.

(continued)

Now let $\lambda > |H_\theta|$ and fix an elementary embedding $j : V \to M$ such that:

- $crit(j) = \kappa$
- $j(\kappa) > \lambda$
- ${}^\lambda M \subset M$
- $j(f)(\kappa) = \dot{\mathbb{P}}$

By the closure condition, which is preserved by $\mathbb{P}_\kappa$, $H_\theta[G] = H_\theta^{M[G]}$, so $\mathbb{P} \in \Gamma^{M[G]}$.

V[G] contains Σ_2-correct filters

V[G] contains Σ_2-correct filters

Proof.

(continued)

Since $j(\mathbb{P}_\kappa)$ is constructed from $j(f)$ as $\mathbb{P}_\kappa$ is from f,

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$$

for some name $\dot{\mathbb{R}}$ for a poset in Γ.

V[G] contains Σ_2-correct filters

Proof.

(continued)

Since $j(\mathbb{P}_\kappa)$ is constructed from $j(f)$ as $\mathbb{P}_\kappa$ is from f,

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$$

for some name $\dot{\mathbb{R}}$ for a poset in Γ.

If $H * K \subseteq \mathbb{P} * \dot{\mathbb{R}}$ is $V[G]$-generic, we can extend j to $j^* : V[G] \to M[G][H][K]$ by

$$j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}.$$

V[G] contains Σ_2-correct filters

Proof.

(continued)

Since $j(\mathbb{P}_\kappa)$ is constructed from $j(f)$ as $\mathbb{P}_\kappa$ is from f,

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$$

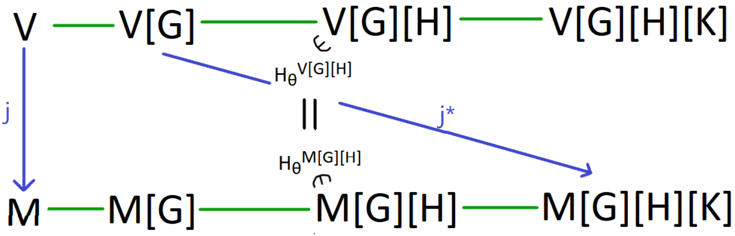
for some name $\dot{\mathbb{R}}$ for a poset in Γ.

If $H * K \subseteq \mathbb{P} * \dot{\mathbb{R}}$ is $V[G]$-generic, we can extend j to $j^* : V[G] \to M[G][H][K]$ by

$$j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}.$$

Since $H \subseteq \mathbb{P}$ is $V[G]$-generic, it meets every dense set in $\mathcal{D}$ and $H_\theta^{V[G][H]} \models \phi(\dot{a}^H)$.

V[G] contains Σ_2-correct filters



V[G] contains Σ_2-correct filters

Proof.
(continued) By the closure conditions, $H_\theta^{V[G][H]} = H_\theta^{M[G][H]}$, so $M[G][H] \models \phi(\dot{a}^H)$. Since $\mathbb{R} \in \Gamma^{M[G][H]}$ and ϕ is provably Γ-persistent, the same holds in $M[G][H][K]$.

V[G] contains Σ_2-correct filters

Proof.

(continued) By the closure conditions, $H_\theta^{V[G][H]} = H_\theta^{M[G][H]}$, so $M[G][H] \models \phi(\dot{a}^H)$. Since $\mathbb{R} \in \Gamma^{M[G][H]}$ and ϕ is provably Γ-persistent, the same holds in $M[G][H][K]$.

Since $|\mathcal{D}| < \kappa = crit(j^*)$ and $\dot{a} \in H_\kappa^{V[G]}$, $j^*(\mathcal{D}) = j^{*}``\mathcal{D}$ and $j^*(\dot{a}) = \dot{a}$. By closure, $j \restriction \dot{\mathbb{P}} \in M$, so $j^* \restriction \mathbb{P}$ and thus $j^{*}``H$ are in $M[G][H][K]$.

V[G] contains Σ_2-correct filters

Proof.
(continued) By the closure conditions, $H_\theta^{V[G][H]} = H_\theta^{M[G][H]}$, so $M[G][H] \models \phi(\dot{a}^H)$. Since $\mathbb{R} \in \Gamma^{M[G][H]}$ and ϕ is provably Γ-persistent, the same holds in $M[G][H][K]$.

Since $|\mathcal{D}| < \kappa = crit(j^*)$ and $\dot{a} \in H_\kappa^{V[G]}$, $j^*(\mathcal{D}) = j^{*}{}''\mathcal{D}$ and $j^*(\dot{a}) = \dot{a}$. By closure, $j \restriction \dot{\mathbb{P}} \in M$, so $j^* \restriction \mathbb{P}$ and thus $j^{*}{}''H$ are in $M[G][H][K]$.

The filter on $j^*(\mathbb{P})$ generated by $j^{*}{}''H$ meets every dense set in $j^{*}{}''\mathcal{D}$, and since none of the conditions relevant to the interpretation of $\dot{a}$ are moved by j^*, it interprets $\dot{a}$ as $\dot{a}^H$.

V[G] contains Σ_2-correct filters

V[G] contains Σ_2-correct filters

Proof.

(continued)

We have therefore shown that

$$M[G][H][K] \models \exists \text{ a filter } F \subseteq j^*(\mathbb{P})\ \forall d \in j^*(\mathcal{D})\ d \cap F \neq \emptyset \wedge \phi(j^*(\dot{a})^F).$$

V[G] contains Σ_2-correct filters

Proof.

(continued)

We have therefore shown that

$$M[G][H][K] \models \exists \text{ a filter } F \subseteq j^*(\mathbb{P})\ \forall d \in j^*(\mathcal{D})\ d \cap F \neq \emptyset \wedge \phi(j^*(\dot{a})^F).$$

Thus by elementarity, in $V[G]$ there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$ such that $\phi(\dot{a}^F)$ holds.

□

Supercompactness for A

Supercompactness for A

We wish to extend the preceding result to formulas of higher complexity. For this, we need the following large cardinal:

Supercompactness for A

We wish to extend the preceding result to formulas of higher complexity. For this, we need the following large cardinal:

Definition

A cardinal κ is said to be ν-supercompact for a class A if there is an elementary embedding $j : V \to M$ such that:

- $crit(j) = \kappa$
- $j(\kappa) > \nu$
- ${}^{\nu}M \subset M$
- $j(A \cap V_\kappa) \cap V_\nu = A \cap V_\nu$

κ is supercompact for A iff it is ν-supercompact for A for all ordinals ν.

Supercompactness for A

We wish to extend the preceding result to formulas of higher complexity. For this, we need the following large cardinal:

Definition

A cardinal κ is said to be ν-supercompact for a class A if there is an elementary embedding $j : V \to M$ such that:

- $crit(j) = \kappa$
- $j(\kappa) > \nu$
- ${}^{\nu}M \subset M$
- $j(A \cap V_\kappa) \cap V_\nu = A \cap V_\nu$

κ is supercompact for A iff it is ν-supercompact for A for all ordinals ν.

We are primarily interested in the case where $A = C^{(n)} := \{\alpha \in Ord \mid V_\alpha \prec_{\Sigma_n} V\}$.

Supercompactness for A

We wish to extend the preceding result to formulas of higher complexity. For this, we need the following large cardinal:

Definition

A cardinal κ is said to be ν-supercompact for a class A if there is an elementary embedding $j : V \to M$ such that:

- $crit(j) = \kappa$
- $j(\kappa) > \nu$
- ${}^{\nu}M \subset M$
- $j(A \cap V_\kappa) \cap V_\nu = A \cap V_\nu$

κ is supercompact for A iff it is ν-supercompact for A for all ordinals ν.

We are primarily interested in the case where
$A = C^{(n)} := \{\alpha \in Ord \mid V_\alpha \prec_{\Sigma_n} V\}$.
Fact: every supercompact cardinal is supercompact for $C^{(1)}$.

Laver functions for A

Laver functions for A

Definition

If κ is supercompact for a class A, $f : \kappa \to V_\kappa$ is a Laver function for A on κ iff for every set x and every ordinal $\nu \geq |trcl(x)|$, there is an elementary embedding $j : V \to M$ witnessing that κ is ν-supercompact for A such that $j(f)(\kappa) = x$.

Laver functions for A

Definition

If κ is supercompact for a class A, $f : \kappa \to V_\kappa$ is a Laver function for A on κ iff for every set x and every ordinal $\nu \geq |trcl(x)|$, there is an elementary embedding $j : V \to M$ witnessing that κ is ν-supercompact for A such that $j(f)(\kappa) = x$.

Lemma

Every cardinal κ supercompact for A has a Laver function for A.

Woodin-Jensen Characterization of Forcing Axioms

Woodin-Jensen Characterization of Forcing Axioms

The following result is helpful in formulating generalized forcing axioms:

Lemma

(Woodin 2010, Jensen 2012)
The following are equivalent for any poset $\mathbb{P}$ *and regular cardinal* $\kappa > \omega_1$*:*

1. $FA_{<\kappa}(\mathbb{P})$
2. *For all cardinals* γ *such that* $\mathbb{P} \in H_\gamma \models ZFC^-$ *and all* $X \subset H_\gamma$ *with* $|X| < \kappa$*, there is a transitive structure* N *with an elementary embedding* $\sigma : N \to H_\gamma$ *such that* $X \cup \{\mathbb{P}\} \subseteq rng(\sigma)$ *and there is an* N*-generic filter* $F \subseteq \sigma^{-1}(\mathbb{P})$*.*

The Σ_n-correct Forcing Axiom

The Σ_n-correct Forcing Axiom

Definition

If Γ is a forcing class, $\kappa > \omega_1$ is a regular cardinal, and n is a positive integer, Σ_n-$CFA_{<\kappa}(\Gamma)$ is the statement that for all posets $\mathbb{P} \in \Gamma$, $\mathbb{P}$-names $\dot{a}$, provably Γ-persistent Σ_n formulas ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, cardinals $\gamma > \kappa$ such that $\dot{a}, \mathbb{P} \in H_\gamma \models ZFC^-$, and $X \subset H_\gamma$ such that $|X| < \kappa$, there is a transitive structure N with an elementary embedding $\sigma : N \to H_\gamma$ such that

- $\dot{a}$, $\mathbb{P}$, and all elements of X are in the range of σ
- $rng(\sigma) \cap \kappa$ is transitive
- there is an N-generic filter $F \subseteq \sigma^{-1}(\mathbb{P})$ such that $\phi(\sigma^{-1}(\dot{a})^F)$ holds.

The Σ_n-correct Forcing Axiom

Definition

If Γ is a forcing class, $\kappa > \omega_1$ is a regular cardinal, and n is a positive integer, Σ_n-$CFA_{<\kappa}(\Gamma)$ is the statement that for all posets $\mathbb{P} \in \Gamma$, $\mathbb{P}$-names $\dot{a}$, provably Γ-persistent Σ_n formulas ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, cardinals $\gamma > \kappa$ such that $\dot{a}, \mathbb{P} \in H_\gamma \models ZFC^-$, and $X \subset H_\gamma$ such that $|X| < \kappa$, there is a transitive structure N with an elementary embedding $\sigma : N \to H_\gamma$ such that

- $\dot{a}$, $\mathbb{P}$, and all elements of X are in the range of σ
- $rng(\sigma) \cap \kappa$ is transitive
- there is an N-generic filter $F \subseteq \sigma^{-1}(\mathbb{P})$ such that $\phi(\sigma^{-1}(\dot{a})^F)$ holds.

Note: Even if $\Delta \subseteq \Gamma$, in general Σ_n-$CFA_{<\kappa}(\Gamma) \not\Rightarrow \Sigma_n$-$CFA_{<\kappa}(\Delta)$

Nice Forcing Classes

Nice Forcing Classes

Definition
(adapted from Aspero and Bagaria (2001))
A forcing class Γ is n-nice iff:

Nice Forcing Classes

Definition

(adapted from Aspero and Bagaria (2001))

A forcing class Γ is *n*-nice iff:

- Γ contains the trivial forcing

Nice Forcing Classes

Definition

(adapted from Aspero and Bagaria (2001))
A forcing class Γ is *n*-nice iff:

- Γ contains the trivial forcing
- Each $\mathbb{P} \in \Gamma$ preserves ω_1

Nice Forcing Classes

Definition

(adapted from Aspero and Bagaria (2001))
A forcing class Γ is *n*-nice iff:

- Γ contains the trivial forcing
- Each $\mathbb{P} \in \Gamma$ preserves ω_1
- If $\mathbb{P} \in \Gamma$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$, then $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$

Nice Forcing Classes

Definition

(adapted from Aspero and Bagaria (2001))
A forcing class Γ is *n*-nice iff:

- Γ contains the trivial forcing
- Each $\mathbb{P} \in \Gamma$ preserves ω_1
- If $\mathbb{P} \in \Gamma$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$, then $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$
- For every inaccessible cardinal κ and every forcing iteration $\langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa\rangle$ of posets in $V_\kappa \cap \Gamma$ with some suitable support, if $\mathbb{P}_\kappa$ is the corresponding limit, then:
  - $\mathbb{P}_\kappa \in \Gamma$
  - $\Vdash_{\mathbb{P}_\alpha} \mathbb{P}_\kappa / \mathbb{P}_\alpha \in \Gamma$ in for all $\alpha < \kappa$
  - If $\mathbb{P}_\alpha \in V_\kappa$ for all $\alpha < \kappa$, then $\mathbb{P}_\kappa$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \kappa \rangle$ and has the κ-cc

Nice Forcing Classes

Definition

(adapted from Aspero and Bagaria (2001))
A forcing class Γ is *n*-nice iff:

- Γ contains the trivial forcing
- Each $\mathbb{P} \in \Gamma$ preserves ω_1
- If $\mathbb{P} \in \Gamma$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$, then $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$
- For every inaccessible cardinal κ and every forcing iteration $\langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ of posets in $V_\kappa \cap \Gamma$ with some suitable support, if $\mathbb{P}_\kappa$ is the corresponding limit, then:
  - $\mathbb{P}_\kappa \in \Gamma$
  - $\Vdash_{\mathbb{P}_\alpha} \mathbb{P}_\kappa / \mathbb{P}_\alpha \in \Gamma$ in for all $\alpha < \kappa$
  - If $\mathbb{P}_\alpha \in V_\kappa$ for all $\alpha < \kappa$, then $\mathbb{P}_\kappa$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \kappa\rangle$ and has the κ-cc
- Γ is Σ_n-definable

Consistency of Σ_n-CFA

Consistency of Σ_n-*CFA*

Theorem

If κ is supercompact for $C^{(n-1)}$ and Γ is an n-nice forcing class, then there is a κ-cc forcing $\mathbb{P}_\kappa \in \Gamma$ such that, if $G \subset \mathbb{P}_\kappa$ is V-generic, $V[G] \models \Sigma_n\text{-}CFA_{<\kappa}(\Gamma)$.

Consistency of Σ_n-*CFA*

Theorem

If κ is supercompact for $C^{(n-1)}$ and Γ is an n-nice forcing class, then there is a κ-cc forcing $\mathbb{P}_\kappa \in \Gamma$ such that, if $G \subset \mathbb{P}_\kappa$ is V-generic, $V[G] \models \Sigma_n\text{-}CFA_{<\kappa}(\Gamma)$.

Proof.

We follow the previous proof closely, noting important differences. Let $f : \kappa \to V_\kappa$ be a Laver function for A, $\mathbb{P}_\kappa$ be the Baumgartner iteration of Γ derived from f and $G \subset \mathbb{P}_\kappa$ be V-generic.

Consistency of Σ_n-*CFA*

Theorem

If κ is supercompact for $C^{(n-1)}$ and Γ is an n-nice forcing class, then there is a κ-cc forcing $\mathbb{P}_\kappa \in \Gamma$ such that, if $G \subset \mathbb{P}_\kappa$ is V-generic, $V[G] \models \Sigma_n\text{-}CFA_{<\kappa}(\Gamma)$.

Proof.

We follow the previous proof closely, noting important differences. Let $f : \kappa \to V_\kappa$ be a Laver function for A, $\mathbb{P}_\kappa$ be the Baumgartner iteration of Γ derived from f and $G \subset \mathbb{P}_\kappa$ be V-generic.

Now, in addition to choosing θ very large, we additionally require $\theta \in C^{(n-1)}$.

Facts about $C^{(n)}$ cardinals

Facts about $C^{(n)}$ cardinals

1: If $\theta \in C^{(n)}$, $\mathbb{Q} \in H_\theta$ is a forcing poset, and $H \subseteq \mathbb{Q}$ is V-generic, $\theta \in (C^{(n)})^{V[H]}$. (Fuchs 2018)

Facts about $C^{(n)}$ cardinals

1: If $\theta \in C^{(n)}$, $\mathbb{Q} \in H_\theta$ is a forcing poset, and $H \subseteq \mathbb{Q}$ is V-generic, $\theta \in (C^{(n)})^{V[H]}$. (Fuchs 2018)

2: If ϕ is a Σ_n formula and b is a parameter, $\phi(b)$ holds iff there is some $\theta \in C^{(n-1)}$ such that $b \in V_\theta$ and $V_\theta \models \phi(b)$.

Consistency of Σ_n-CFA

Consistency of Σ_n-CFA

Proof.

(continued)

Therefore we can find a $\theta > \gamma$ such that $V_\theta^{V[G]} \models \mathbb{P} \in \Gamma$ and, if $H \subseteq \mathbb{P}$ is $V[G]$-generic, $V_\theta^{V[G][H]} \models \phi(\dot{a}^H)$.

Consistency of Σ_n-*CFA*

Proof.

(continued)

Therefore we can find a $\theta > \gamma$ such that $V_\theta^{V[G]} \models \mathbb{P} \in \Gamma$ and, if $H \subseteq \mathbb{P}$ is $V[G]$-generic, $V_\theta^{V[G][H]} \models \phi(\dot{a}^H)$.

As before, we choose $\lambda > |V_\theta| (= \theta)$ and let $j : V \to M$ witness that κ is λ-supercompact for $C^{(n-1)}$ such that $j(f)(\kappa) = \dot{\mathbb{P}}$.

Consistency of Σ_n-*CFA*

Proof.

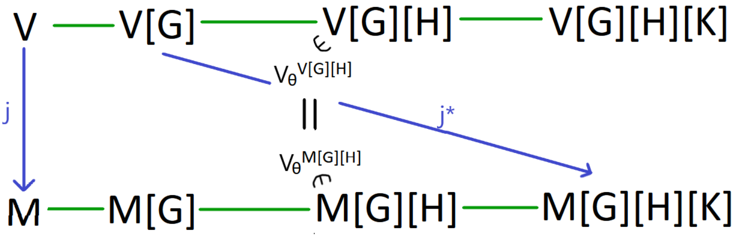
(continued)

Therefore we can find a $\theta > \gamma$ such that $V_\theta^{V[G]} \models \mathbb{P} \in \Gamma$ and, if $H \subseteq \mathbb{P}$ is $V[G]$-generic, $V_\theta^{V[G][H]} \models \phi(\dot{a}^H)$.

As before, we choose $\lambda > |V_\theta| (= \theta)$ and let $j : V \to M$ witness that κ is λ-supercompact for $C^{(n-1)}$ such that $j(f)(\kappa) = \dot{\mathbb{P}}$.

By elementarity, $j(C^{(n-1)} \cap V_\kappa) = (C^{(n-1)})^M \cap j(\kappa)$, so $\theta \in C^{(n-1)} \cap \lambda = j(C^{(n-1)} \cap V_\kappa) \cap V_\lambda = (C^{(n-1)})^M \cap \lambda$. Therefore we can use V_θ to transport the truth of $\mathbb{P} \in \Gamma$ and $\phi(\dot{a}^H)$ from forcing extensions of V to forcing extensions of M.

Consistency Diagram



Consistency of Σ_n-CFA

Consistency of Σ_n-*CFA*

Proof.

(continued)

As before, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$, so we let $K \subseteq \mathbb{R}$ be $V[G][H]$-generic and extend j to $j^* : V[G] \to M[G][H][K]$, where $j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}$.

Consistency of Σ_n-CFA

Proof.

(continued)

As before, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$, so we let $K \subseteq \mathbb{R}$ be $V[G][H]$-generic and extend j to $j^* : V[G] \to M[G][H][K]$, where $j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}$.

Now let $\sigma' : H_\gamma^{V[G]} \to H_{j(\gamma)}^{M[G][H][K]}$ be the restriction of j^*. Since $j \restriction H_\gamma^V \in M$, $\sigma' \in M[G][H][K]$.

Consistency of Σ_n-*CFA*

Proof.

(continued)

As before, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}} * \dot{\mathbb{R}}$, so we let $K \subseteq \mathbb{R}$ be $V[G][H]$-generic and extend j to $j^* : V[G] \to M[G][H][K]$, where $j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}$.

Now let $\sigma' : H_\gamma^{V[G]} \to H_{j(\gamma)}^{M[G][H][K]}$ be the restriction of j^*. Since $j \restriction H_\gamma^V \in M$, $\sigma' \in M[G][H][K]$.

Since $|X| < \kappa = crit(j^*)$, $j^*(X) = j^{*\prime\prime} X$, so $j^*(X) \subset rng(\sigma')$. Finally, since $crit(\sigma') = \kappa$ gets mapped to $j(\kappa)$, $rng(\sigma') \cap j^*(\kappa) = \kappa$ is transitive.

Consistency of Σ_n-*CFA*

Proof.

(continued)

Therefore we have shown:

Consistency of Σ_n-CFA

Proof.
(continued)
Therefore we have shown:

$M[G][H][K] \models$ "there exists a transitive structure N (i.e. $H_\gamma^{V[G]}$)
with an elementary embedding $\sigma' : N \to H_{j^*(\gamma)}$ such that
$j^*(X) \cup \{j^*(\dot{a}), j^*(\mathbb{P})\} \subset rng(\sigma'), rng(\sigma') \cap j^*(\kappa)$ is transitive, and
there is an N-generic filter $H \subseteq \sigma'^{-1}(j^*(\mathbb{P}))$ such that $\phi(\sigma'^{-1}(j^*(\dot{a}))^H)$."

By elementarity, in $V[G]$ we have a transitive N with an elementary embedding $\sigma : N \to H_\gamma^{V[G]}$ such that $X \cup \{\dot{a}, \mathbb{P}\} \subset rng(\sigma)$, $rng(\sigma) \cap \kappa$ is transitive, and there is an N-generic filter $F \subseteq \sigma^{-1}(\mathbb{P})$ such that $\phi(\sigma^{-1}(\dot{a})^F)$. □

Weak Genericity

Weak Genericity

To formulate bounded versions of Σ_n-correct forcing axioms, we need the following bounded version of being generic over a small structure:

Weak Genericity

To formulate bounded versions of Σ_n-correct forcing axioms, we need the following bounded version of being generic over a small structure:

Definition

If β is an ordinal, $N \models ZFC^- \wedge$ "β is a cardinal" is transitive, and $\mathbb{P}$ is a complete Boolean algebra in N, a filter $F \subseteq \mathbb{P}$ is $< \beta$-weakly N-generic iff for every maximal antichain of $\mathbb{P}$ $A \in N$ with $|A|^N < \beta$, $A \cap F \neq \emptyset$.

Σ_n-correct Bounded Forcing Axioms

Σ_n-correct Bounded Forcing Axioms

Definition

Σ_n-$CBFA^{<\lambda}_{<\kappa}(\Gamma)$, for Γ a forcing class, n a positive integer, and $\lambda \geq \kappa > \omega_1$ cardinals, is the statement that for all complete Boolean algebras $\mathbb{P}$, $\mathbb{P}$-names $\dot{a} \in H_\lambda$, cardinals $\gamma \geq \lambda$ such that $\mathbb{P} \in H_\gamma \models ZFC^-$, $X \subset H_\gamma$ with $|X| < \kappa$, and provably Γ-persistent Σ_n formulas ϕ such that $\Vdash_{\mathbb{P}} \phi(\dot{a})$, there is a transitive structure N with an elementary embedding $\sigma : N \to H_\gamma$ such that $X \cup \{\mathbb{P}, \dot{a}, \kappa, \lambda\} \subseteq rng(\sigma)$ and a $< \bar{\lambda}$-weakly N-generic filter $F \subseteq \bar{\mathbb{P}}$ such that $\phi(\bar{a}^F)$ holds, where $\bar{\mathbb{P}} := \sigma^{-1}(\mathbb{P})$, $\bar{\lambda} := \sigma^{-1}(\lambda)$, and $\bar{a} := \sigma^{-1}(\dot{a})$.

Σ_n-correctly H_λ-reflecting cardinals

Σ_n-correctly H_λ-reflecting cardinals

The consistency of Σ_n-$CBFA^{<\lambda}_{<\kappa}(\Gamma)$ for n-nice Γ and regular κ and λ is derived from the following large cardinal (modeled on Miyamoto 1998):

Σ_n-correctly H_λ-reflecting cardinals

The consistency of Σ_n-$CBFA^{\leq\lambda}_{<\kappa}(\Gamma)$ for n-nice Γ and regular κ and λ is derived from the following large cardinal (modeled on Miyamoto 1998):

Definition

For cardinals κ and λ, we say that κ is Σ_n-correctly H_λ-reflecting iff κ is regular and for every Σ_n formula ϕ and $a \in H_\lambda$, if $\phi(a)$ holds, then the set of $Z \prec H_\lambda$ of size less than κ and containing a such that $V_\kappa \models \phi(\pi_Z(a))$ (where π_Z is the Mostowski collapse map for Z) is stationary in $[H_\lambda]^{<\kappa}$. If $\lambda = \kappa^{+\alpha}$, we say that κ is Σ_n-correctly $+\alpha$ reflecting.

Correctly Reflecting Laver Functions

Correctly Reflecting Laver Functions

Definition

A function $g : \kappa \to V_\kappa$ is called a Σ_n-correctly H_λ-reflecting Laver function on κ if for all Σ_n formulas ϕ and $a \in H_\lambda$ such that $\phi(a)$, there are stationarily many $Z \prec H_\lambda$ of size less than κ such that $V_\kappa \models \phi(\pi_Z(a))$ and $g(\pi_Z(\kappa)) = \pi_Z(a)$.

Correctly Reflecting Laver Functions

Definition

A function $g : \kappa \to V_\kappa$ is called a Σ_n-correctly H_λ-reflecting Laver function on κ if for all Σ_n formulas ϕ and $a \in H_\lambda$ such that $\phi(a)$, there are stationarily many $Z \prec H_\lambda$ of size less than κ such that $V_\kappa \models \phi(\pi_Z(a))$ and $g(\pi_Z(\kappa)) = \pi_Z(a)$.

Lemma

If κ is Σ_n-correctly H_λ-reflecting, the fast function forcing $\mathbb{F}_\kappa$ adds a correctly reflecting Laver function on κ.

Standard Forcing Axioms are Σ_1-correct

Standard Forcing Axioms are Σ_1-correct

Theorem

(Miyamoto 1998 for proper forcing)
For any forcing class Γ *and regular cardinals* $\lambda \geq \kappa > \omega_1$,
BFA$^{\leq\lambda}_{<\kappa}(\Gamma)$ *is equivalent to* Σ_1-*CBFA*$^{\leq\lambda}_{<\kappa}(\Gamma)$.

Standard Forcing Axioms are Σ_1-correct

Theorem

(Miyamoto 1998 for proper forcing)
For any forcing class Γ and regular cardinals $\lambda \geq \kappa > \omega_1$, $BFA^{\leq\lambda}_{<\kappa}(\Gamma)$ is equivalent to Σ_1-$CBFA^{\leq\lambda}_{<\kappa}(\Gamma)$.

Proof sketch: First prove a bounded version of the Woodin-Jensen characterization using weak genericity. Then for any suitable ϕ, $\mathbb{P}$, X, $\dot{a}$, γ, $\Vdash_{\mathbb{P}} \phi(\dot{a})$ is absolute to H_γ. If N is transitive and elementarily embeds into H_γ and $F \subseteq \bar{\mathbb{P}}$ is $< \bar{\lambda}$-weakly N-generic, $N^{\bar{\mathbb{P}}}/F \models \phi([\bar{a}]_F)$, $[\bar{a}]_F$ is in the well-founded part of $N^{\bar{\mathbb{P}}}/F$, and it collapses to $\bar{a}^F$, where bars denote the inverse of the embedding $N \to H_\gamma$.

Maximality Principles

Maximality Principles

Definition

(Stavi and Vaananen 2002, Hamkins 2003)
For Γ a forcing class, n a positive integer, and S a class of parameters, Σ_n-$MP_\Gamma(S)$ is the assertion that for all provably Γ-persistent Σ_n formulas ϕ and $a \in S$ such that there is a $\mathbb{P} \in \Gamma$ which forces $\phi(a)$, then $\phi(a)$ already holds in the ground model.

Σ_n-MP is Equivalent to Symmetric Σ_n-*CBFA*

Σ_n-MP is Equivalent to Symmetric Σ_n-*CBFA*

Theorem

(Bagaria 2000 for $n = 1$)
For any positive integer n and regular cardinal κ, Σ_n-$MP_\Gamma(H_\kappa)$ is equivalent to Σ_n-$CBFA^{<\kappa}_{<\kappa}(\Gamma)$.

Σ_n-MP is Equivalent to Symmetric Σ_n-*CBFA*

Theorem

(Bagaria 2000 for $n = 1$)
For any positive integer n and regular cardinal κ, Σ_n-$MP_\Gamma(H_\kappa)$ is equivalent to Σ_n-$CBFA^{<\kappa}_{<\kappa}(\Gamma)$.

Proof sketch: ($\Leftarrow$): Use check names

Σ_n-MP is Equivalent to Symmetric Σ_n-*CBFA*

Theorem

(Bagaria 2000 for $n = 1$)
For any positive integer n and regular cardinal κ, Σ_n-$MP_\Gamma(H_\kappa)$ is equivalent to Σ_n-$CBFA^{<\kappa}_{<\kappa}(\Gamma)$.

Proof sketch: ($\Leftarrow$): Use check names
($\Rightarrow$): Construct a suitable $N \in H_\kappa$ with an elementary embedding $\sigma : N \to H_\gamma$ that fixes $\dot{a}$, then consider the statement "there exists a $< \bar{\kappa}$-weakly N-generic filter $F \subseteq \bar{\mathbb{P}}$ such that $\phi(\dot{a}^F)$ holds".

Applications

Applications

Proposition

If Γ *is the class of countably closed forcings or of subcomplete forcings, then* Σ_2*-*$CBFA^{<\omega_2}_{<\omega_2}(\Gamma)$ *implies:*

a) $\diamondsuit$

b) $\neg KH$

Applications

Proposition

If Γ *is the class of countably closed forcings or of subcomplete forcings, then* Σ_2*-CBFA*$^{<\omega_2}_{<\omega_2}(\Gamma)$ *implies:*
a) $\diamondsuit$
b) $\neg KH$

Proof.

a) $\diamondsuit$ is a Σ_2 sentence which is forced by the countably closed poset $Add(\omega_1, 1)$, and it is preserved by all subcomplete forcing.

Applications

Proposition

If Γ is the class of countably closed forcings or of subcomplete forcings, then Σ_2-CBFA$^{<\omega_2}_{<\omega_2}(\Gamma)$ implies:
a) $\diamondsuit$
b) $\neg KH$

Proof.

a) $\diamondsuit$ is a Σ_2 sentence which is forced by the countably closed poset $Add(\omega_1, 1)$, and it is preserved by all subcomplete forcing.
b) If T is any ω_1-tree, subcomplete forcing does not add branches to it, and there is a countably closed forcing which collapses the cardinality of its branches to ω_1. "T is not a Kurepa tree" is then a Σ_2 formula which can be made true by countably closed forcing and is preserved under arbitrary subcomplete forcing, so it is true in V. □

Applications

Applications

Proposition

If Γ *is any nonempty subclass of proper forcing,* Σ_2*-CBFA*$^{<\lambda}_{<\kappa}(\Gamma)$ *(with the added condition that* $|N| < \kappa$*) implies that for all cardinals* θ *(regular) and* ν *with* $\nu < \kappa$ *and* $\theta^{\omega\cdot\nu} < \lambda$*,* $X \subset \theta$ *smaller than* κ*, and sequences* $\mathcal{S} = \langle S_\beta \mid \beta < \nu \rangle$ *of stationary subsets of* $[\theta]^\omega$*, then there is a* $Y \subset \theta$ *such that* $X \subseteq Y$*,* $|Y| < \kappa$*, and* $S_\beta \cap [Y]^\omega$ *is stationary in* $[Y]^\omega$ *for all* $\beta < \nu$*.*

Applications

Proposition

If Γ is any nonempty subclass of proper forcing, Σ_2-$CBFA^{<\lambda}_{<\kappa}(\Gamma)$ (with the added condition that $|N| < \kappa$) implies that for all cardinals θ (regular) and ν with $\nu < \kappa$ and $\theta^{\omega\cdot\nu} < \lambda$, $X \subset \theta$ smaller than κ, and sequences $\mathcal{S} = \langle S_\beta \mid \beta < \nu\rangle$ of stationary subsets of $[\theta]^\omega$, then there is a $Y \subset \theta$ such that $X \subseteq Y$, $|Y| < \kappa$, and $S_\beta \cap [Y]^\omega$ is stationary in $[Y]^\omega$ for all $\beta < \nu$.

Proof sketch: Take $\sigma : N \to H_\gamma$ with $rng(\sigma) \cap H_\kappa$ transitive, $X \cup \{\mathcal{S}, \theta, \nu\} \subseteq rng(\sigma)$, and $\sigma^{-1}(\mathcal{S})$ a sequence of ν stationary subsets of $[\bar{\theta}]^\omega$, where $\bar{\theta} := \sigma^{-1}(\theta)$, since stationarity is Π_1 and preserved by proper forcing. Let $Y = \sigma"\bar{\theta}$.

It can be shown that $\sigma"\sigma^{-1}(a) = a$ for all $a \in [\theta]^\omega \cap rng(\sigma)$ and $\sigma"\sigma^{-1}(S_\beta) = S_\beta \cap rng(\sigma) \subseteq S_\beta \cap [Y]^\omega$. To show that the latter is stationary, fix $\beta < \nu$ and $h : [Y]^{<\omega} \to Y$, and let $h' = \sigma^{-1} \circ h \circ \sigma$.

Applications

Applications

Proof sketch (continued): By the stationarity of $\sigma^{-1}(S_\beta)$, there is an $\bar{a} \in \sigma^{-1}(S_\beta)$ closed under h', so $a := \sigma(\bar{a}) \in \sigma``\sigma^{-1}(S_\beta)$, and it is easy to see that $h``a \subseteq a$. Thus $\sigma``\sigma^{-1}(S_\beta)$ is stationary in $[Y]^\omega$, so $S_\beta \cap [Y]^\omega$ is as well. $\square$

References

- David Aspero and Joan Bagaria. "Bounded Forcing Axioms and the Continuum". In: *Annals of Pure and Applied Logic* 109 (2001), pp. 179-203.
- Joan Bagaria. "Bounded Forcing Axioms as Principles of Generic Absoluteness". In: *Archive for Mathematical Logic* 39 (2000), pp. 393–401.
- Gunter Fuchs. "Subcomplete Forcing Axioms and Definable Well-Orders". 2018. doi: 10.48550/arXiv.1708.08167. url: https://arxiv.org/abs/1708.08167v2
- Joel David Hamkins. "A Simple Maximality Principle". In: *The Journal of Symbolic Logic* 68.2 (June 2003), pp. 527–550. doi: 10.2178/jsl/1052669062
- Ronald B. Jensen. *Forcing Axioms Compatible with CH*. Scanned handwritten notes. 2012? url: https://www.mathematik.hu-berlin.de/~raesch/org/jensen.html.
- Tadatoshi Miyamoto. "A Note on Weak Segments of PFA". In: *Proceedings of the Sixth Asian Logic Conference: Beijing, China, 20-24 May 1996* (1998), pp. 175–197
- Jonathan Stavi and Jouko Väänänen. "Reflection Principles for the Continuum". In: *Logic and Algebra*. Ed. by Yi Zhang. Vol. 302. Contemporary Mathematics. American Mathematical Society, 2002, pp. 59–84. isbn: 978-0-8218-2984-4. doi: http://dx.doi.org/10.1090/conm/302.
- W. Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. Second revised. de Gruyter Series in Logic and Its Applications 1. De Gruyt, 2010. isbn: 978-3-11-019702-0.