Vopěnka's Alternative Set Theory and its mathematical context

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Plan for talk

Why this historical work?
– Complement Vopěnka's own reflection on the AST; and that of the group.

- introduction to AST
- main concepts and axioms
- mathematical origins
- philosophical considerations

'AST' will be used as an acronym for 'Alternative Set Theory'.

AST — briefly in data

Location: **Prague**; Bratislava.

Earlier: **The Theory of Semisets** (with P. Hájek) **1972**

2 books:

- **Mathematics in the Alternative Set Theory**, *Teubner Verlag*, **1979**
- **An Introduction to Mathematics in the Alternative Set Theory**, *Alfa, Bratislava*, **1989** (in Slovak)

Journal papers
mostly available from the **Czech Digital Mathematical Library** www.dml.cz

Vopěnka's AST seminar: contributions by
K. Čuda, J. Mlček, A. Sochor, A. Vencovská, J. Chudáček, M. Resl, K. Trlifajová, B. Vojtášková, J. Sgall, J. Witzany, J. Guričan, M. Kalina, P. Zlatoš and others.

Later: **Discourses with Geometry**, **1989** onwards (in Czech);
further historical work, etc. . . .

NB. **New Infinitary Mathematics** is due April 2023 in *Karolinum*.

Vopěnka's views on set theory

Set theory is primarily a **theory of infinity**.

In particular, Cantor's set theory develops infinity according to Cantor's ideas:

- actual infinity (iterative constructions are completed);
- "Cantorian finitism"(Hallett's term): infinite sets behave just as finite sets do (ordinal and cardinal numbers, power set axiom, choice, ...).

[Hallett: Cantorian set theory and the limitations of size. Clarendon Press, 1986]

NB. "Cantor's set theory" is used broadly. For example,

- the Theory of Semisets, or
- Quine's New Foundations

have been, at some point, listed under (or close to) Cantor's set theory.

Cantor's set theory depends on formal means. (Independent statements.)

"At present, no reasons for the acceptance of a nontrivial theory of infinity are known. All such theories must be speculative in character. Consequently, their results mentioning infinite cardinalities will be vacuous if their speculative background is rejected."
[Teubner, p. 51]

Cantor's set theory has become the dominant mathematical ontology.
It employs far more superstructure than needed.

Why Alternative Set Theory?

Vopěnka aims at a mathematical theory of natural infinity.

AST rejects actual infinity and in particular, actually infinite sets.
All sets are "classically" finite.

Sets may represent "exact knowledge".

Proper classes are (potentially) infinite.
Infinity manifests itself as an absence of easy survey.
"Our infinity is a phenomenon occurring when we observe large sets."
[Teubner, p. 35]

Infinity as **indefiniteness.**

Classes may represent "point of view".

Sochor: difference between sets and classes was key to Vopěnka in the early seventies.
A. Sochor: Petr Vopěnka ($*$ 16.5.1935). APAL 109, 2001.

AST had foundational ambitions:
develop enough mathematics and to compete with Cantor's set theory.

Axioms for sets:

- (extensionality for sets) $\forall xy\,[(x = y) \equiv \forall z(z \in x \equiv z \in y)]$;
- (empty set) $\exists x \forall Y\,(Y \notin x)$;
- (set successor) $\forall xy\,\exists z\,(z = x \cup \{y\})$;
- (induction) $\varphi(\emptyset)\,\&\,\forall xy\,[\varphi(x) \rightarrow \varphi(x \cup \{y\})] \rightarrow \forall x \varphi(x)$;
- (regularity) $\exists x\,\varphi(x) \rightarrow \exists x(\varphi(x)\,\&\,\forall y \in x \neg\varphi(y))$,

for φ a set formula.

This theory is equivalent to $\mathrm{ZF}_{\mathrm{fin}}$ (i.e., $\mathrm{ZF} \setminus \{\mathrm{Inf}\} \cup \{\neg\mathrm{Inf}\}$), with regularity schema as above.

In particular, all sets in the AST are clasically finite.

In fact, AST is a conservative extension of $\mathrm{ZF}_{\mathrm{fin}}$.

[Sochor: Metamathematics of the AST III, Comment. Mathematicae Universitatis Carolinae 24, 1983, §9]

Natural numbers in the AST

A set is a natural number iff is transitive and totally ordered by $\in$.

$\mathbb{N}$ denotes the proper class of natural numbers.

For each x there is (unique) $\alpha \in \mathbb{N}$ s.t. $x \hat{\approx} \alpha$ (**set** bijection).

Operations:

- $S(\alpha) = \alpha \cup \{\alpha\}$;
- $\alpha + \beta = \gamma$ iff $\gamma \hat{\approx} \alpha \cup (\{\beta\} \times \beta)$;
- $\alpha \cdot \beta = \gamma$ iff $\gamma \hat{\approx} \alpha \times \beta$.

Natural order: $\alpha \leq \beta$ iff $\alpha \in \beta \vee \alpha = \beta$.

Induction: $\varphi(0) \,\&\, \forall \alpha \in \mathbb{N}\, [\varphi(\alpha) \to \varphi(S(\alpha))] \to \forall \alpha \in \mathbb{N}\, \varphi(\alpha)$.
for φ set formula.

$\mathbb{N}$ with S, $+$ and $\cdot$ interprets PA.

Axioms for classes

(extensionality for classes) $\forall X, Y[(X = Y) \equiv \forall Z(Z \in X \equiv Z \in Y)]$

(class comprehension) $\exists Y \forall x\, (x \in Y \equiv \Phi(x))$
for **every** $\Phi(x)$ not containing Y.

A class X is finite iff all subclasses are sets: $\mathrm{Fin}(X)$ iff $\forall Y \subseteq X(\mathrm{Set}(Y))$.
Finite classes are sets; $\mathrm{Fin} = \{x \in V \mid \mathrm{Fin}(x)\}$.

A semiset is a subclass of a set.

(existence of proper semisets) $\exists X \exists y(X \subseteq y \,\&\, \neg\mathrm{Set}(X))$.
Hence infinite sets exist.

A is well-ordered by a relation R provided R is linear on A and each nonemepty $B \subseteq A$ has R-first element.

(choice) There is a well-ordering of V.

Intermezzo: the notion of semiset

Semisets occur already in the development of the **theory of semisets**, which is based on NBG.

Axiomatization includes: NBG for one sort of variables (X^0) and a weak fragment (TC) for another sort (X) subsuming the other sort. (And sets.)
Then proper semisets can exist.

The collection of infinite natural numbers is already used as example of semiset.

[Hájek: Why semisets? Comment. Mathematicae Universitatis Carolinae 14(3), 1973]

Finite natural numbers in AST

$\mathbb{FN} = \{\alpha \in \mathbb{N} : \mathrm{Fin}(\alpha)\}$ (finite natural numbers).

Since $\forall x \exists \alpha \in \mathbb{N}(x \hat{\approx} \alpha)$, infinite natural numbers exist.

$\mathbb{FN}$ is a proper initial segment of $\mathbb{N}$ (nat. order), and of any infinite $\alpha \in \mathbb{N}$.

$\mathbb{FN}$ is not a set (closed under successors; no last element in nat. order).

$\mathbb{FN}$ is a prototypical semiset.

$\mathbb{FN}$ is closed under S, $+$, and $\cdot$.

Induction for $\mathbb{FN}$:
$\Phi(0)$ and $\forall n \in \mathbb{FN}\, [\Phi(n) \to \Phi(n \cup \{n\})]$ implies $\forall n \in \mathbb{FN}\, \Phi(n)$
Φ any formula.

$\mathbb{FN}$, with operations, interprets PA.

More axioms for classes

(prolongation) Let F be a class function on $\mathbb{FN}$. Then there is a set function $f \supseteq F$.

Existence of proper semisets follows.

A class X is countable if $X \approx n$ or $X \approx \mathbb{FN}$.

A countably infinite class is a proper semiset.
Thus the universal class V is not countable.

(cardinalities) if X and Y are uncountable, then $X \approx Y$.
Lemma: each countably infinite class is a semiset.

(induction for formal set formulas):
For any formula φ $\quad V \models \varphi(\emptyset) \,\&\, \forall xy\, [\varphi(x) \rightarrow \varphi(x \cup \{y\})] \rightarrow \forall x \varphi(x)$

Some of Vopěnka's comments on prolongation

"People have always tried to go beyond the horizon; this is a typical human aspiration. The aim is not merely to shift the horizon further away but to transcend it in the mind. Mathematics is one of the most important instruments for this; it formulates exact statements which transcend the framework of perception.
[. . .]

[. . .]
The prolongation axiom is a hypothesis which serves as a base for **exact knowledge exceeding evidence**."

[Teubner, p. 41]

Extensions of AST

Let FV denote the class of hereditarily finite sets.

(axiom of elementary equivalence) FV is elementarily equivalent to V.

The axiom is independent of the AST (as given above).
[Sochor: Metamathematics of the AST III, 1983]

For example, we have $\mathrm{ZF_{fin}} \nvdash \mathrm{Con}(\overline{\mathrm{ZF_{fin}}})$ unless $\mathrm{ZF_{fin}}$ is inconsistent.
Hence we also have $\mathrm{AST} \nvdash \mathrm{Con}(\overline{\mathrm{ZF_{fin}}})$ by conservativity.

On the other hand, we have $\mathrm{AST} \vdash_F \mathrm{Con}^F(\overline{\mathrm{ZF}^F_{\mathrm{fin}}})$.
[Sochor: Metamathematics of the AST I, 1979]

Bimodal logic of provability in the AST was studied by Jeřábek
[Jeřábek: Provability logic in the AST. Charles University in Prague, 2001]

Models of the AST in ZF

Theorem [Sochor]
In ZF, consider a consistent theory $T \supseteq \mathrm{ZF_{fin}}$ in the language of sets. There is a model M of the AST such that the set reduct of M validates T.

Proof: Let $M \models T$.
Let $M' = (V^*, \in^*)$ be the ultrapower of M over a nontrivial ultrafilter on ω.
Add to M' each subset $X \subseteq V^*$ unless there is $x \in V^*$ s.t. $X = \{y \mid (V^*, \in^*) \models y \in x$ (for AST-classes).
(Assume CH to cater for the AST-axiom of cardinalities)
This yields a model M'' of AST, while the set part validates T by Łoś theorem.

Corollary. AST is conservative over $\mathrm{ZF_{fin}}$ for set formulas.

Proof: Let φ be a set formula not provable in $\mathrm{ZF_{fin}}$, i.e., $\mathrm{ZF_{fin}} + \neg\varphi$ is consistent.
By Theorem above, we obtain a model M of the AST s.t. $M \models \neg\varphi$; so AST $\nvdash \varphi$.

[Sochor, Metamathematics of the AST III, 1982]

Pudlák and Sochor discuss expandability of models of $\mathrm{ZF_{fin}}$ to models of AST.

[Pudlák, Sochor: Models of the Alternative Set Theory. JSL 49, 1984]

Rieger's works

Skolem **1934**: nonstandard model of PA.

[Th. Skolem: Über de Nichtcharakterisierbarkeit der Zahlenreihe [...]. Fundamenta Mathematicae 23, 1934]

[Rieger: A contribution to Gödel's axiomatic set theory, II. Czechoslovak mathematical journal 9, 1959.]

Models the theory $\mathrm{NBG}_{\mathrm{fin}}$ (i.e., NBG — as in Gödel 1940 — with infinity replaced by its negation) with "dyadic integral numbers": binary expansions of the form $\Sigma_{i=k}^{\infty} c_i 2^i$ for some $k \in \mathrm{Z}$, and with $c_i \in \{0, 1\}$.

(Expanding Ackermann's interpretation to classes: $x \in y$ iff x-th digit of bin. exp. of y is 1. Sets are represented by "nonnegative integers".)

This is referred to as the standard model.

He then provides axiomatic theory of "s-t-rings" with example as above. Skolem's construction (iterated) can be used to obtain model of $\mathrm{NBG}_{\mathrm{fin}}$ with uncountably many natural numbers.

Nonstandard universes

Vopěnka frequented Rieger's seminar at some point and learnt about Skolem's construction from Rieger.

"At that time [when?], I also began to frequent Rieger's seminar in mathematical logic, where I obtained the knowledge of Skolem's non-standard model of arithmetic of natural numbers. Thus armed, I created a non-standard model of Gödel–Bernays set theory (that is, a model with non-standard natural numbers) in 1961, using the ultraproduct method (instead of an ultrafilter, it uses a maximal ideal)."

[Vopěnka: Prague set theory seminar. In *Witnessed years: Essays in Honour of Petr Hájek*, 2010]

1961 onwards: Vopěnka provided a nonstandard models of NBG;
(i.e., intp. of NBG in itself, with nonstandard natural numbers).

[Vopěnka: Odin metod postroenia nestandardnoi modeli aksiomaticheskoi teorii mnozhestv Bernaysa-Gödela. Doklady Akademii nauk SSSR, 143(1), 1961]

Robinson's nonstandard analysis

[Robinson: Non-standard analysis. Proc. Royal Acad. Sci., 1961]
[Robinson: Non-standard Analysis. North Holland, 1966]

Rieger's report on the Infinitistic methods symposium:
"On September 2 – September 8, 1959, Warsaw hosted an international symposium in the foundations of mathematics on *infinitistic methods* of mathematical logic. [. . .] Many presentations were dedicated to non-normal [i.e., nonstandard] models of arithmetic of natural numbers and of axiomatic set theory."
[L.S. Rieger: Report on international symposium on foundations of mathematics. Časopis pro pěstování matematiky, 1960] (in Czech)

Rieger (and Vopěnka?) worked with nonstandard universes before Robinson's NSA was published.

NSA encouraged Vopěnka to work with (axiomatize) nonstandard structures.

AST is an (early) axiomatic approach to nonstandard analysis.

Axiomatic method and new ontology

Vopěnka refers explicitly to NSA:
"From the formal and technical point of view, Alternative Set Theory is rather near to Nonstandard Analysis and can be considered, from this point of view, for a particular case of Nonstandard Analysis."
[Teubner, p. 3]

Vopěnka's **axiomatization** of nonstandard analysis.

[Hájek: Why Semisets. CMUC 14, 1973]
[Pudlák: Logical Foundations of Mathematics and Computational Complexity: A Gentle Introduction. Springer 2013, p. 237]

Vopěnka has little interest in models of AST in ZFC.

Instead, AST provides an ontology for mathematics:

- only elements of $\mathbb{N}$ play the role of natural numbers in the universe of sets;
- in a limit universe, the role of standard natural numbers is played by $\mathbb{FN}$;
- in a witnessed universe the classical natural numbers correspond to elements of $\mathbb{N}$ whereas $\mathbb{FN}$ forms the canonical representative of the way to the horizon.

[Teubner, p. 63]

Rational numbers in the AST

The integers are built from natural numbers:
$\mathbb{N}^- = \mathbb{N} \cup \{\langle 0, \alpha \rangle | \alpha \neq 0\}$; analogously for $\mathbb{FN}^-$.

Then $\mathbb{RN}$ is defined as the quotient field of $\mathbb{N}^-$, and analogously for $\mathbb{FRN}$.

Rationals x, y are infinitely near, $x \doteq y$ iff

- $|x - y| < 1/n$ for each nonzero $n \in \mathbb{FN}$, or
- $n < x$ and $n < y$ for each $n \in \mathbb{FN}$, or
- $x < -n$ and $y < -n$ for each $n \in \mathbb{FN}$.

A rational x is infinitely small iff $x \doteq 0$.

Reciprocal infinities ("distance" vs. "depth")

Real numbers are the quotient of rational numbers by $\doteq$.

Nonstandard axiomatic set theories

Two nonstandard set theories close to ZFC:

[Nelson: Internal set theory: A new approach to nonstandard analysis. Bull. AMS, 1977]

[Hrbacek: Nonstandard set theory. Amer. Math. Monthly, 1979]

"These three initial attempts to fully axiomatize nonstandard mathematics can hardly be linearly ordered in any reasonable sense, with any sort of preference assigned in some sound manner. It is fair to assert, on the base of available records, that all three were undertaken independently of each other and led to results of comparable quality (although not of comparable impact on the practice of nonstandard mathematics, where IST has preference), in addition all three were based upon earlier development in foundations of nonstandard analysis."

[Kanovei, Reeken: Nonstandard Analysis, Axiomatically. Springer, 2004]

Feasibility considerations (down the ages)

"From two integers k, l one passes immediately to k^l; this process leads in a few steps to numbers which are far larger than any occurring in experience, e.g., $67^{257^{729}}$. // Intuitionism, like ordinary mathematics, claims that this number can be represented by an Arabic numeral. Could not one press further the criticism which intuitionism makes of existential assertions and raise the question: What does it mean to claim the existence of an Arabic numeral for the foregoing number, since in practice we are not in a position to obtain it? Brouwer appeals to intuition, but one can doubt that the evidence for it really is intuitive. Isn't this rather an application of the general method of analogy, consisting in extending to inaccessible numbers the relations which we can concretely verify for accessible numbers? As a matter of fact, the reason for applying this analogy is strengthened by the fact that there is no precise boundary between the numbers which are accessible and those which are not."

[Bernays: Platonism in mathematics. 1935]

Feasibility considerations (down the ages) – cont'd

"Let us consider the series F of feasible numbers, i.e., of those up to which it is possible to count. The number 0 is feasible and if n is feasible then [...] n' also is feasible. And each feasible number can be obtained from 0 by adding $'$; so F forms a natural number series. But 10^{12} does not belong to F."

[Alexander Yessenin-Volpin: The ultra-intuitionistic criticism and the antitraditional program for the foundations of mathematics. *Intuitionism and Proof Theory.* North Holland, 1970, 3–45]

Witnessed universes in the AST

A theory of a witnessed universe imposes a semiset within a concrete set.

"The theory of witnessed universes is in fact inconsistent in the classical sense. If c is an entirely concrete set (say, the set of all natural numbers less than $67^{293^{159}}$) then it can be obtained in finitely many steps from the empty set by successive addition of single elements; thus c is finite. On the other hand if c has a proper subsemisets then c is infinite in our sense. But our proof that c is finite has itself infinitely many steps (in our sense). [...]"
[Teubner, p. 37]

NB.
[R. Solovay: Interpretability in set theories. Letter to Petr Hájek, www.cs.cas.cz/hajek/RSolovayZFGB.pdf]

AST did not develop the theory of witnessed universes.

Parikh's almost consistent theories

Let **T** be **PA** and $\mathbf{T}^+$ its extension by def.'s of p.r.f.'s.
Let F be a new unary predicate symbol, with axioms:

(a1) $\mathrm{F}(0)$;

(a2) $\mathrm{F}(x) \to \mathrm{F}(S(x))$;

(a3) $\mathrm{F}(x) \,\&\, y < x \to \mathrm{F}(y)$; and

(a4) $\neg\mathrm{F}(\theta)$ for some closed p.r. term θ (in the language of $\mathbf{T}^+$).

Let $\mathbf{T}_1^+$ be $\mathbf{T}^+ \cup \{(a1)–(a4)\}$; then $\mathbf{T}_1^+$ is inconsistent.

Parikh however obtains an "almost consistency" result. For a proof P in $\mathbf{T}_1^+$ he introduces some "proof complexity measures", say $\bar{k}(P) = \langle k_i(P)\rangle_{i \leq n}$ (e.g., there is an $i \leq n$ for which $k_i(P)$ is the number of instances of (a2) in P).

Theorem: [Parikh] there is a primitive recursive function g such that if φ is a formula in the language of T^+ (i.e., F does not occur in φ), P is a proof of φ in $\mathbf{T}_1^+$, and the value of θ in $\mathbb{N}$ is $m > g(\bar{k}(P))$, then $\mathbf{T}^+ \vdash \varphi$.

Hájek constructs a semiset of feasible numbers in TSS (almost consistently).
[Hájek: Why Semisets? Comment. Mathematicae Universitatis Carolinae, 1973]

Modelling the sorites, and Dean's paper

AST presentations often refer to soritical situations ("small numbers" etc.)
As Dean points out, it offers a (consistent) model for vagueness (in terms of limit universes).

[Walter Dean: Strict finitism, feasibility, and the sorites. Rev. Symb. Logic 11(2), 2018]

"[...] feasibilism is unlike most traditional approaches to vagueness in that it seeks to provide a proof-theoretic rather than a model-theoretic account of the meaning of vague predicates. For since theories like $\mathbf{T}_1^+$ are inconsistent in the 'in principle' sense of ('classical') proof theory, they also do not possess interpretations in the sense of ('classical') model theory."

V^i are theories of weak two-sorted arithmetic (following Zambella).

Proposition [Dean] For all i, we have $V^i \nvdash \forall X \exists y \mathrm{Bin}(X, y)$
(presuming V^i consistent).

Recap – timeline

1934 Skolem's nonstandard model of PA
1957 – 1963 Rieger publishes a series of papers on NBG
1959 symposium Infinitistic Methods in Warsaw
1961 Robinson's paper on NSA
1961 Vopěnka's nonstandard model of NBG
1963 – 1968 Vopěnka Prague school of (classical) set theory
1968 Soviet invasion to Czechoslovakia
1972 The Theory of Semisets published by North Holland
1970's and 1980's seminar on AST at Faculty of Mathematics and Physics
1973 Vopěnka circulates notes on AST
1975 (onwards) Sochor publishes papers on AST in English
1979 Mathematics in the Alternative Set Theory published by Teubner
1968 – 1980 Polívka employed at Faculty of Mathematics and Physics
1989 Introduction to mathematics in the Alternative Set Theory published by Alfa (in Slovak)
1989 1st Symposium on the Alternative Set Theory held in Stará Lesná
1989 velvet revolution in Czechoslovakia
1990 Vopěnka appointed full professor of mathematics at Faculty of Mathematics and Physics. Shortly afterwards becomes Minister of Education.
2003 – 2015 Vopěnka works at the University of West Bohemia in Pilsen
2015 New Infinitary Mathematics published (in Czech)

Concluding remarks

AST presents a coherent new development of infinity ("lack of an easy survey"), different from the Cantorian one.

Inspired by nonstandard models and methods (Skolem, Robinson, ...),
to some extent also by feasibility considerations (Parikh,...).
Provably consistent in ZFC.

(But) presented as new ontology for mathematics, with axiomatic framework, without substantial support of ZFC or nonstandard structures developed in it.

The axiomatization was intended as an open system.
Exposition of **mathematics in the AST**.

Neither the Theory of Semisets nor the AST became widely known.

AST was developed as theory of limit universes.
Witnessed universes just mentioned in passing.

Vopěnka not known to **compare the AST to other foundational efforts** (e.g., other nonstandard set theories, reverse mathematics, or predicative mathematics).

References

not given (properly) in the text:

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