MULTI-LEVEL NONSTANDARD ANALYSIS, THE AXIOM OF CHOICE, AND RECENT WORK OF R. JIN

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INTRODUCTION.

Nonstandard Analysis is sometimes criticized for its implicit dependence on the Axiom of Choice (**AC**).
(Bishop, Connes,..)

Indeed, model-theoretic frameworks for nonstandard methods require the existence of nonprincipal ultrafilters over $\mathbb{N}$, a strong form of **AC**.

If $*$ is the mapping that assigns to each $X \subseteq \mathbb{N}$ its nonstandard extension $^*X$, and if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ is an unlimited integer, then the set $U = \{X \subseteq \mathbb{N} \mid \nu \in {}^*X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

Axiomatic frameworks (**IST**, **HST**,...) cannot avoid the dependence on **AC** by simply removing it from the list of axioms.

These theories postulate some version of **Standardization Principle**:

For every formula Φ *in the language of the theory (possibly with parameters) and every standard set* A *there exists a standard set* S *such that for all standard* x*,*

$$x \in S \iff x \in A \wedge \Phi(x).$$

This set is denoted ${}^{\mathrm{st}}\{x \in A \mid \Phi(x)\}$.

It follows that, for an unlimited $\nu \in \mathbb{N}$, the standard set $U = {}^{\mathrm{st}}\{X \in \mathcal{P}(\mathbb{N}) \mid \nu \in X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

Hence all results obtained by nonstandard analysis in these frameworks depend on the Axiom of Choice.

While strong forms of **AC**, such as Zorn's Lemma, are instrumental in many abstract areas of mathematics, such as general topology (the product of compact spaces is compact), measure theory (there exist sets that are not Lebesgue measurable) or functional analysis (Hahn-Banach theorem), it is undesirable to have to rely on them for results in "ordinary" mathematics such as calculus, finite combinatorics and number theory.

In the paper

KH and Mikhail G. Katz,
Infinitesimal analysis without the Axiom of Choice,
Ann. Pure Appl. Logic 172, 6 (2021)
https://doi.org/10.1016/j.apal.2021.102959
https://arxiv.org/abs/2009.04980

we have formulated a set theory **SPOT** in the st-$\in$-language.

SPOT *is a conservative extension of* **ZF**. Arguments carried out in **SPOT** thus do not depend on any form of **AC**.

To avoid **AC**, Standardization has to be weakened.

Infinitesimal analysis can be carried out in **SPOT**.

Traditional proofs in "ordinary" mathematics either do not use **AC** at all, or refer only to its weak forms, notably the Axiom of Countable Choice (**ACC**) or the stronger Axiom of Dependent Choice (**ADC**). These axioms are generally accepted and often used without comment.

These weak forms are necessary to prove eg. the equivalence of the ε-δ definition and the sequential definition of continuity for functions $f : X \subseteq \mathbb{R} \to \mathbb{R}$, or the countable additivity of Lebegue measure, but they do not imply the strong consequences of **AC** such as the existence of nonprincipal ultrafilters or the Banach–Tarski paradox.

The theory **SCOT** is a strengthening of **SPOT** by **ADC**.

SPOT and **SCOT**

By an $\in$-language we mean the language that contains a binary membership predicate $\in$ and is enriched by defined symbols for constants, relations, functions and operations customary in traditional mathematics.

For example, it contains names $\mathbb{N}$ and $\mathbb{R}$ for the sets of natural and real numbers; these sets are viewed as defined in the traditional way ($\mathbb{N}$ is the least inductive set, $\mathbb{R}$ is defined in terms of Dedekind cuts or Cauchy sequences).

Nonstandard set theories add to the $\in$-language a unary predicate symbol st, where $\mathrm{st}(x)$ reads "x is standard," and possibly other symbols.

They postulate that standard infinite sets contain also nonstandard elements. For example, $\mathbb{R}$ contains infinitesimals and unlimited reals, and $\mathbb{N}$ contains unlimited natural numbers.

The axioms of **SPOT** are:

ZF (Zermelo - Fraenkel Set Theory)

T (Transfer) Let ϕ be an $\in$-formula with standard parameters. Then

$$\forall^{\mathrm{st}} x\ \phi(x) \Rightarrow \forall x\ \phi(x).$$

O (Nontriviality) $\quad \exists \nu \in \mathbb{N}\ \forall^{\mathrm{st}} n \in \mathbb{N}\ (n \neq \nu).$

SP (Standard Part)

$$\forall A \subseteq \mathbb{N}\ \exists^{\mathrm{st}} B \subseteq \mathbb{N}\ \forall^{\mathrm{st}} n \in \mathbb{N}\ (n \in B \ \leftrightarrow\ n \in A).$$

The axiom **SP** is often used in the equivalent form

$$\forall x \in \mathbb{R}\ (x \text{ limited } \Rightarrow \exists^{\mathrm{st}} r \in \mathbb{R}\ (x \simeq r)) \qquad (\mathbf{SP}')$$

where x is *limited* iff $|x| \leq n$ for some standard $n \in \mathbb{N}$, and $x \simeq r$ iff $|x - r| \leq 1/n$ for all standard $n \in \mathbb{N}$, $n \neq 0$; x is *infinitesimal* if $x \simeq 0 \ \wedge \ x \neq 0$.
The unique standard real number r is called the *standard part of x* or the *shadow of x*; notation $r = \mathsf{sh}(x)$.

The axiom **SP** is also equivalent to Standardization over countable sets for $\in$-formulas (with arbitrary parameters).

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- assignable vs. inassignable distinction
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- assignable vs. inassignable distinction [standard vs. nonstandard]
- law of continuity [Transfer]
- existence of infinitesimals [Nontriviality]
- equality up to infinitesimal terms that need to be discarded [Standard Part].

Some of the general results provable in **SPOT** are:

Proposition. *Standard natural numbers precede all nonstandard ones:* $\forall^{\mathrm{st}} n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (m < n \Rightarrow \mathrm{st}(m))$.

Proposition. (Countable Idealization)
Let ϕ be an $\in$-formula with arbitrary parameters.

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists x\ \forall m \in \mathbb{N}\ (m \leq n\ \Rightarrow \phi(m, x))\ \leftrightarrow\ \exists x\ \forall^{\mathrm{st}} n \in \mathbb{N}\ \phi(n, x).$$

The theory **SPOT** proves an important stronger version of **SP**.

Definition. An st-$\in$-formula $\Phi(v_1, \ldots, v_r)$ is $\mathrm{st}_{\mathbb{N}}$-*prenex* if it is of the form

$$\mathsf{Q}^{\mathrm{st}}_{\mathbb{N}} u_1 \ldots \mathsf{Q}^{\mathrm{st}}_{\mathbb{N}} u_s\, \psi(u_1, \ldots, u_s, v_1, \ldots, v_r)$$

where ψ is an $\in$-formula, each Q stands for $\exists$ or $\forall$, and $\forall^{\mathrm{st}}_{\mathbb{N}}\, u \ldots$, $\exists^{\mathrm{st}}_{\mathbb{N}}\, u \ldots$ are shorthand for respectively $\forall^{\mathrm{st}} u\, (u \in \mathbb{N} \Rightarrow \ldots)$ and $\exists^{\mathrm{st}} u\, (u \in \mathbb{N} \wedge \ldots)$.

Proposition. (Countable Standardization for $\mathrm{st}_{\mathbb{N}}$-Prenex Formulas)
Let Φ *be an* $\mathrm{st}_{\mathbb{N}}$*-prenex formula with arbitrary parameters. Then*

$$\exists^{\mathrm{st}} S \, \forall^{\mathrm{st}} n \, (n \in S \leftrightarrow n \in \mathbb{N} \wedge \Phi(n)).$$

Of course, $\mathbb{N}$ can be replaced by any standard countable set.

It is useful to add two additional special cases of Standardization.

SN (Standardization for formulas with no parameters)
Let Φ *be an* st-$\in$*-formula with* standard *parameters. Then*
$\forall^{\mathrm{st}} A\, \exists^{\mathrm{st}} S\, \forall^{\mathrm{st}} x\, (x \in S \leftrightarrow x \in A \wedge \Phi(x))$.

SF (Standardization over standard finite sets)
Let Φ *be an* st-$\in$*-formula with arbitrary parameters. Then*
$\forall^{\mathrm{st\,fin}} A\, \exists^{\mathrm{st}} S\, \forall^{\mathrm{st}} x\, (x \in S \leftrightarrow x \in A \wedge \Phi(x))$.

SPOT$^+$ is **SPOT** + **SN** + **SF**.

Theorem 1

SPOT$^+$ is a conservative extension of **ZF**.

The theory **SCOT** is **SPOT**$^+$ + **DC**, where

DC (Dependent Choice for st-$\in$-formulas):
Let $\Phi(u, v)$ be an st-$\in$-formula with arbitrary parameters. If B is a set, $b \in B$ and $\forall x \in B \, \exists y \in B \, \Phi(x, y)$, then there is a sequence $\langle b_n \mid n \in \mathbb{N} \rangle$ such that $b_0 = b$ and $\forall^{\mathrm{st}} n \in \mathbb{N} \, (b_n \in B \wedge \Phi(b_n, b_{n+1}))$.

One consequence is

SC (Countable Standardization)
Let Ψ be an st-$\in$-formula with arbitrary parameters. Then $\exists^{\mathrm{st}} S \, \forall^{\mathrm{st}} n \, (n \in S \leftrightarrow n \in \mathbb{N} \wedge \Psi(n))$.

Theorem 2

SCOT is a conservative extension of **ZF** + **ADC**.

It allows such features as an infinitesimal construction of the Lebesgue measure.

MANY LEVELS OF STANDARDNESS.

Theories with many levels of standardness have been developed in
Y. Péraire, *Théorie relative des ensembles internes*, Osaka J. Math. 29 (1992), 267–297 (**RIST**)
and
KH, *Relative set theory: Internal view,* Journal of Logic and Analysis 1:8 (2009), 1–108. (**GRIST**).

The characteristic feature of these theories is that the unary standardness predicate $\mathrm{st}(v)$ is subsumed under the binary *relative standardness* predicate $\mathrm{sr}(u, v)$.

For example, the nonstandard definition of the derivative

$$f'(a) = \mathsf{sh}\, \frac{f(a+h) - f(a)}{h} \quad \text{where } h \text{ is infinitesimal,}$$

which in a single-level nonstandard analysis works for standard f and a only, in these theories works for all f and a, provided "infinitesimal" is understood as "infinitesimal relative to the level of f and a" and "sh" is "sh relative to the level of f and a."

In the book KH, O. Lessmann and R. O'Donovan, *Analysis using Relative Infinitesimals*, Chapman and Hall, 2015, 316 pp. this apporach is used to develop elementary calculus.

Nonstandard analysis with multiple levels of standardness has been used in combinatorics and number theory by Terence Tao, Renling Jin, Mauro Di Nasso and others.

Renling Jin recently gave a groundbreaking nonstandard proof of Szemerédi's Theorem in a model-theoretic framework that has three levels of infinity.

Szemerédi's Theorem:
If $D \subseteq \mathbb{N}$ has a positive upper density, then D contains a k-term arithmetic progression for every $k \in \mathbb{N}$

R. Jin, *A simple combinatorial proof of Szemerédi's theorem via three levels of infinities*
Discrete Analysis, 2023:15, 27 pp.
`https://arXiv.org/abs/2203.06322v1`
and the Editorial Introduction at
`https://discreteanalysisjournal.com/article/87772-a-simple-combinatorial-proof-of-szemeredi-s-t`

Jin's work using multi-level nonstandard analysis goes beyond the features postulated by **RIST** and **GRIST** in that it also employs nontrivial elementary embeddings (ie, other than those provided by inclusion of one level in a higher level).

ULTRAFILTERS, ULTRAPOWERS, AND THEIR ITERATIONS.

In this section we work in **ZFC**.
U is an ultrafilter over I.
For $f, g \in \mathbb{V}^I$ define
$f =_U g$ iff $\{i \in I \mid f(i) = g(i)\} \in U$,
$f \in_U g$ iff $\{i \in I \mid f(i) \in g(i)\} \in U$.
$[f]_U$ is the equivalence class of f modulo $=_U$ (use Scott's Trick),
$\mathbb{V}^I/U = \{[f]_U \mid f \in \mathbb{V}^I\}$, and $[f]_U \in_U [g]_U$ iff $f \in_U g$.
The *ultrapower* of $\mathbb{V}$ by U is the structure $(\mathbb{V}^I/U, \in_U)$.

Let $\pi : I \to J$. Define the ultrafilter $V = \pi[U]$ over J by
$\pi[U] = \{Y \subseteq J \mid \pi^{-1}[Y] \in U\}$.
Define $\widetilde{\pi} : \mathbb{V}^J/V \to \mathbb{V}^I/U$ by $\widetilde{\pi}([g]_V) = [g \circ \pi]_U$.

$\widetilde{\pi}$ is an elementary embedding of $\mathbb{V}^J/V$ into $\mathbb{V}^I/U$.
(Łoś's Theorem)

The *tensor product* of ultrafilters U and V, respectively over I and J, is the ultrafilter over $I \times J$ defined by
$Z \in U \otimes V$ iff $\{x \in I \mid \{y \in J \mid \langle x, y \rangle \in Z\} \in V\} \in U$.
(Note the order!)

The *n-th tensor power* of U is the ultrafilter over I^n defined by recursion:
$\otimes^0 U = \{\{\emptyset\}\}; \quad \otimes^1 U = U; \quad \otimes^{n+1} U = U \otimes (\otimes^n U)$.

a, b range over finite subsets of $\mathbb{N}$.
If $|a| = n$, let π be the mapping of I^n onto I^{a} induced by the order-preserving mapping of n onto a.
$U_{\mathrm{a}} = \pi[\otimes^n U]$ is an ultrafilter over I^{a}.

For a $\subseteq$ b let $\pi^{\mathrm{b}}_{\mathrm{a}}$ be the restriction map of I^{b} onto I^{a}:
$\pi^{\mathrm{b}}_{\mathrm{a}}(\mathbf{i}) = \mathbf{i} \restriction \mathrm{a}$ for $\mathbf{i} \in I^{\mathrm{b}}$
It is easy to see that $U_{\mathrm{a}} = \pi^{\mathrm{b}}_{\mathrm{a}}[U_{\mathrm{b}}]$.
Hence $\widetilde{\pi}^{\mathrm{b}}_{\mathrm{a}}$ is an elementary embedding of $\mathbb{V}^{I^{\mathrm{a}}}/U_{\mathrm{a}}$ into $\mathbb{V}^{I^{\mathrm{b}}}/U_{\mathrm{b}}$.
If $|\mathrm{a}| = |\mathrm{b}|$ then $\widetilde{\pi}^{\mathrm{b}}_{\mathrm{a}}$ is an isomorphism of $\mathbb{V}^{I^{\mathrm{a}}}/U_{\mathrm{a}}$ and $\mathbb{V}^{I^{\mathrm{b}}}/U_{\mathrm{b}}$.

The *limit ultrapower* of $\mathbb{V}$ by U is the limit of the directed system of structures $\mathbb{V}^{I^{\mathrm{a}}}/U, \quad \widetilde{\pi}^{\mathrm{b}}_{\mathrm{a}}; \quad \mathrm{a}, \mathrm{b} \in \mathcal{P}^{fin}(\omega), \mathrm{a} \subseteq \mathrm{b}$.

SPOTS

The language of **SPOTS** has a binary predicate symbol $\in$, a binary predicate symbol sr [$\mathsf{sr}(u,v)$ reads "v is *u-standard*"] and a ternary function symbol ir that captures the relevant isomorphisms. The unary predicate $\mathtt{st}(v)$ stands for for $\mathsf{sr}(\emptyset, v)$,

We use the class notation
$\mathbb{S}_{\mathsf{a}} = \{x \mid \mathsf{sr}(\mathsf{a},x)\}$ and $\mathbb{I}_{\mathsf{a}}^{\mathsf{b}} = \{\langle x,y\rangle \mid \mathsf{ir}(\mathsf{a},\mathsf{b},x) = y\}$
and refer to the subscripts and superscripts as *labels*.

Symbols a, b (with decorations) are used *exclusively* as variables for labels. They are intended to range over *standard finite* subsets of $\mathbb{N}$.

For $n \in \mathbb{N} \cap \mathbb{S}_0$ we call $\mathbb{S}_n$ the *n-th level of standardness*.

We let $r \oplus \mathrm{a} = \{r + s \mid s \in \mathrm{a}\}$;

For any natural number r let $\Phi^{\uparrow r}$ be the formula obtained from Φ by shifting all labels by r; i.e., by replacing each occurrence of every $\mathbb{S}_{\mathrm{a}}$ with $\mathbb{S}_{r\oplus\mathrm{a}}$ and each occurrence of $\mathbb{I}_{\mathrm{a}}^{\mathrm{b}}$ with $\mathbb{I}_{r\oplus\mathrm{a}}^{r\oplus\mathrm{b}}$.

$\mathrm{a} < \mathrm{b}$ stands for $\forall s \in \mathrm{a}\ \forall t \in \mathrm{b}\ (s < t)$.

The iterated ultrapower construction described above suggests the axioms **IS**, **GT**, **HO** and **EE**:

IS (Structural axioms)
For all standard finite a, a′

$$\mathsf{a} \subseteq \mathsf{a}' \Rightarrow \mathbb{S}_{\mathsf{a}} \subseteq \mathbb{S}_{\mathsf{a}'}.$$

For all standard finite a, a′, a″ of the same cardinality

$$\mathbb{I}_{\mathsf{a}}^{\mathsf{a}'} : \mathbb{S}_{\mathsf{a}} \to \mathbb{S}_{\mathsf{a}'}, \quad \mathbb{I}_{\mathsf{a}}^{\mathsf{a}} = Id_{\mathbb{S}_{\mathsf{a}}}, \quad \mathbb{I}_{\mathsf{a}'}^{\mathsf{a}} = (\mathbb{I}_{\mathsf{a}}^{\mathsf{a}'})^{-1}, \quad \mathbb{I}_{\mathsf{a}}^{\mathsf{a}'} \circ \mathbb{I}_{\mathsf{a}'}^{\mathsf{a}''} = \mathbb{I}_{\mathsf{a}}^{\mathsf{a}''};$$

$$\forall x, z \in \mathbb{S}_{\mathsf{a}}\ (x \in z \ \leftrightarrow\ \mathbb{I}_{\mathsf{a}}^{\mathsf{a}'}(x) \in \mathbb{I}_{\mathsf{a}}^{\mathsf{a}'}(z));$$

For all standard finite a, a′ of the same cardinality and b $\subset a$

$$x \in \mathbb{S}_{\mathsf{b}} \Rightarrow \mathbb{I}_{\mathsf{b}}^{\mathsf{b}'}(x) = \mathbb{I}_{\mathsf{a}}^{\mathsf{a}'}(x)$$

where b′ is the image of b by the order-preserving map of a onto a′.

GT (Generalized Transfer)
Let ϕ be an $\in$-formula. Then for all standard finite a

$$\forall x_1, \ldots, x_k \in \mathbb{S}_a \ (\forall x \in \mathbb{S}_a \ \phi(x, x_1, \ldots, x_k) \Rightarrow \forall x \ \phi(x, x_1, \ldots, x_k)).$$

GT (Generalized Transfer)
Let ϕ be an $\in$-formula. Then for all standard finite a

$$\forall x_1, \ldots, x_k \in \mathbb{S}_a \ (\forall x \in \mathbb{S}_{\mathrm{a}} \ \phi(x, x_1, \ldots, x_k) \Rightarrow \forall x \ \phi(x, x_1, \ldots, x_k)) \, .$$

HO (Homogeneous Shift)
Let Φ be any formula. Then for any natural number r and all standard finite a

$$\forall \bar{x} \in \mathbb{S}_{\mathrm{a}} \, [\, \Phi(\bar{x}) \ \leftrightarrow \ \Phi^{\uparrow r}(\mathbb{I}_{\mathrm{a}}^{r \oplus \mathrm{a}}(\bar{x}))\,].$$

HO is justified by the *Factoring Lemma*:

$$(\mathbb{V}^{I^{r+n}}/U_{r+n},\ \in_{U_{r+n}},\ \mathbb{V}^{I^{r\oplus \mathrm{a}}}/U_{r\oplus \mathrm{a}},\ \widetilde{\pi}^{r\oplus \mathrm{b}}_{r\oplus \mathrm{a}};\ \mathrm{a},\mathrm{b}\subset n)$$

is canonically isomorphic to

$$(\mathbb{V}^{I^{n}}/U_{n}\ \in_{U_n},\mathbb{V}^{I^{\mathrm{a}}}/U_{\mathrm{a}},\widetilde{\pi}^{\mathrm{b}}_{\mathrm{a}};\ \mathrm{a},\mathrm{b}\subset n)^{I^r}/U_r.$$

The last axiom asserts that the natural numbers at level n are an end extension of the natural numbers at levels $m < n$ and more.

EE (End Extension) For all standard finite a

$$\forall n \in \mathbb{S}_{\mathsf{a}} \cap \mathbb{N}\, (n \in \mathbb{S}_0 \lor \forall \mathsf{b} < \mathsf{a}\, \forall m \in \mathbb{S}_{\mathsf{b}}\, (m < n)).$$

In other words, every natural number $n \in \mathbb{S}_{\mathsf{a}}$ is either standard or greater than every natural number at levels less than $\min \mathsf{a}$.

SPOTS is the theory **SPOT**$^+$ + **IS** + **GT** + **HO** + **EE**.
SCOTS is the theory **SCOT** + **IS** + **GT** + **HO** + **EE** = **SPOTS** + **DC**.
[Nontriviality is modified to $\exists \nu \in \mathbb{N}\ (\mathrm{sr}_1(\nu)\ \wedge\ \forall^{\mathrm{st}} n \in \mathbb{N}\ (n \neq \nu)$
and **SF**, **DC** admit all formulas in the language of **SPOTS**.]

Theorem 3

SCOTS is a conservative extension of **ZF** + **ADC**.

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Conjecture

SPOTS is a conservative extension of **ZF**.

JIN'S PROOF OF RAMSEY'S THEOREM IN SPOTS

Ramsey's Theorem

Given a coloring $c : [\mathbb{N}]^n \to r$ for some $n, r \in \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $c \upharpoonright [H]^n$ is a constant function.

I formalize in **SPOTS** the proof presented by Renling Jin in his talk at the conference *Logical methods in Ramsey Theory and related topics*, Pisa July 9 - 11, 2023

`https://events.dm.unipi.it/event/151/`

It suffices to prove the theorem under the assumption that n, r, c are standard; the general result then follows by Transfer.

Let $\mathbb{I} = \mathbb{I}_{\{0,1,\dots,n-1\}}^{\{1,2,\dots,n\}}$. Fix $\nu \in \mathbb{N} \cap (\mathbb{S}_1 \setminus \mathbb{S}_0)$.
Define the n-tuple $\bar{x} = \langle x_1, \dots, x_n \rangle$ by
$x_1 = \nu, \quad x_{i+1} = \mathbb{I}(x_i)$ for $i = 1, 2, \dots, n-1$
[this is justified by **SF**].

Let $c_0 = c(\bar{x})$.

Define a strictly increasing sequence $\{a_m\}_{m=1}^{?} \subseteq \mathbb{N}$ recursively (notation $A_m = \{a_1, \ldots, a_m\}$):
$a_{m+1} =$ the least $a \in \mathbb{N}$ s.t. $a > a_m \wedge c \restriction [A_m \cup \{a\} \cup \bar{x}]^n = c_0$ if such a exists.

Let $A = \bigcup_{m=1}^{?} A_m$. Then A is an (internal) set and by **SP** there is a standard set H such that $\forall^{\mathrm{st}} \bar{z}\, (\bar{z} \in H \leftrightarrow \bar{z} \in A)$.
Clearly $c \restriction [H]^n = c_0$.

It remains to prove that H is infinite, i.e., that a_m is defined and standard for all standard $m \in \mathbb{N}$.

Fix a standard $m \in \mathbb{N}$. The sentence
$\exists x \in \mathbb{N} \cap \mathbb{S}_1 \ (x > a_m \wedge c \restriction [A_m \cup \{x, \mathbb{I}(x_1), \ldots, \mathbb{I}(x_{n-1})\}]^n = c_0)$
is true (just let $x = x_1$).

By **HO**
$\exists x \in \mathbb{N} \cap \mathbb{S}_0 \ (x > a_m \wedge c \restriction [A_m \cup \{x, x_1, \ldots, x_{n-1}\}]^n = c_0)$.
We let a_{m+1} be the least such x and note that it is standard.

We have $c \restriction [A_{m+1} \cup \{x_1, \ldots, x_{n-1}\}]^n = c_0$.
It remains to show that $c \restriction [A_{m+1} \cup \{x_1, \ldots, x_{n-1}, x_n\}]^n = c_0$.

Consider $\bar{b} = \{b_1 < \ldots < b_n\} \in [A_{m+1} \cup \{x_1, \ldots, x_{n-1}, x_n\}]^n$.
If $b_n < x_n$ then $b_n \leq x_{n-1}$ and $c(\bar{b}) = c_0$.
If $b_1 = x_1$ then $\bar{b} = \bar{x}$ and $c(\bar{b}) = c(\bar{x})) = c_0$.
Otherwise $b_1 \in \mathbb{N} \cap \mathbb{S}_0$ and $b_n = x_n$.

Let p be the largest value such that $x_p \notin \bar{b}$; , $1 \leq p < n$.

Let $\mathbb{J} = \mathbb{I}^{\{0,\ldots,p-1,p+1,\ldots,x_n\}}_{\{0,\ldots,n-1\}}$.

We note that
$\mathbb{J}(b_j) = b_j$ for $j \leq p$;
$b_j = x_j$ and $\mathbb{J}(b_j) = \mathbb{J}(x_j) = x_{j+1}$ for $p < j \leq n-1$
(because $\mathbb{I}^{\{p+1,\ldots,n\}}_{\{p,\ldots,n-1\}} \subseteq \mathbb{I}, \mathbb{J}$, i.e., $\mathbb{I}$ and $\mathbb{J}$ agree on $\mathbb{S}_{\{p,\ldots,n-1\}}$).

Let $\bar{b}' = \mathbb{J}^{-1}(\bar{b})$.
Then $\bar{b}' \in [A_{m+1} \cup \{x_1, \ldots, x_{n-1}\}]^n$, hence $c(\bar{b}') = c_0$.
By **HO** shift via $\mathbb{J}$, $c(\bar{b}) = c_0$. □

JIN'S PROOF OF SZEMEREDI'S THEOREM IN SPOTS

Jin's proof uses four universes ($\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$ and $\mathbb{V}_3$) and some additional elementary embeddings.
Let $\mathbb{N}_j = \mathbb{N} \cap \mathbb{V}_j$ and $\mathbb{R}_j = \mathbb{R} \cap \mathbb{V}_j$ for $j = 0, 1, 2, 3$.

Jin's *Property 3.1* summarizes what is required.

0. $\mathbb{V}_0 \prec \mathbb{V}_1 \prec \mathbb{V}_2 \prec \mathbb{V}_3$.
1. $\mathbb{N}_{j+1}$ is an end extension of $\mathbb{N}_j$ $(j = 0, 1, 2)$.
2. For $j' > j$, Countable Idealization holds from $\mathbb{V}_j$ to $\mathbb{V}_j'$:
Let ϕ be an $\in$-formula with parameters from $\mathbb{V}_{j'}$.
$\forall n \in \mathbb{N}_j \, \exists x \, \forall m \in \mathbb{N} \, (m \leq n \Rightarrow \phi(m, x)) \leftrightarrow \exists x \, \forall n \in \mathbb{N}_j \, \phi(n, x)$.
3.There is an elementary embedding i_* of $(\mathbb{V}_2; \mathbb{R}_0, \mathbb{R}_1)$ to $(\mathbb{V}_3; \mathbb{R}_1, \mathbb{R}_2)$.
4. There is an elementary embedding i_1 of $\mathbb{V}_1$ to $\mathbb{V}_2$ such that $i_1 \upharpoonright \mathbb{N}_0$ is an identity map and $i_1(a) \in \mathbb{N}_2 \setminus \mathbb{N}_1$ for each $a \in \mathbb{N}_1 \setminus \mathbb{N}_0$.
5.There is an elementary embedding i_2 of $\mathbb{V}_2$ to $\mathbb{V}_3$ such that $i_2 \upharpoonright \mathbb{N}_1$ is an identity map and $i_2(a) \in \mathbb{N}_3 \setminus \mathbb{N}_2$ for each $a \in \mathbb{N}_2 \setminus \mathbb{N}_1$.

Proposition

SPOTS interprets Jin's Property 3.1.

Proof. We define:
$\mathbb{V}_0 = \mathbb{S}_0$, $\mathbb{V}_1 = \mathbb{S}_{\{0\}}$, $\mathbb{V}_2 = \mathbb{S}_{\{0,1\}}$, $\mathbb{V}_3 = \mathbb{S}_{\{0,1,2\}}$, and
$i_1 = \mathbb{I}^{\{1\}}_{\{0\}}$, $i_2 = \mathbb{I}^{\{0,2\}}_{\{0,1\}}$, $i_* = \mathbb{I}^{\{1,2\}}_{\{0,1\}}$. □

The only remaining issue is the use of Standardization. **SPOT** proves the existence of densities as defined by Renling Jin.

Strong Upper Banach Densities. In our notation:
For finite $A \subseteq \mathbb{N}$ with $|A|$ unlimited, the *strong upper Banach density of A* is defined by

$$\mathrm{SD}(A) = \sup{}^{\mathrm{st}}\{\mathrm{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited}\}.$$

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If $S \subseteq \mathbb{N}$ has $\mathrm{SD}(S) = \eta \in \mathbb{R}$ (note η is standard) and $A \subseteq S$, the *strong upper Banach density of A relative to S* is defined by $\mathrm{SD}_S(A) =$

$$\sup{}^{\mathrm{st}}\{\mathrm{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited} \wedge \mathrm{sh}(|S \cap P|/|P|) = \eta\}.$$

Proof. SPOT does not prove the existence of the standard sets whose sup needs to be taken, but we rewrite the definition of $\mathrm{SD}_S(A)$ as follows:

$\mathrm{SD}_S(A) = \sup{}^{\mathrm{st}}\{q \in \mathbb{Q} \mid \Phi(q)\}$ where $\Phi(q)$ is the formula

$$\exists P\,[\forall^{st}_{\mathbb{N}} i\,(|P| > i) \wedge \forall^{st}_{\mathbb{N}} j(|\,|S \cap P|/|P| - \eta| < \tfrac{1}{j+1}) \wedge q \leq |A \cap P|/|P|\,].$$

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The formula Φ is equivalent to

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which, upon the exchange of the order of $\exists P$ and $\forall^{st}_{\mathbb{N}} i$, enabled by Countable Idealization, converts to an $\mathrm{st}_{\mathbb{N}}$-prenex formula

$$\forall^{st}_{\mathbb{N}} i\, \exists P\, [\,(|P| > i) \wedge (|\,|S \cap P|/|P| - \eta| < \tfrac{1}{i+1}) \wedge q \leq |A \cap P|/|P|\,],$$

CONSERVATIVITY

These results are established by constructions that extend and combine the methods of forcing developed by Ali Enayat and Mitchell Spector.

A. Enayat, *From bounded arithmetic to second order arithmetic via automorphisms*, in: A. Enayat, I. Kalantari, and M. Moniri (Eds.), Logic in Tehran, Lecture Notes in Logic, vol. 26, ASL and AK Peters, 2006
`http://academic2.american.edu/~enayat`

M. Spector, *Extended ultrapowers and the Vopěnka–Hrbáček theorem without choice*, Journal of Symboic Logic 56, 2 (1991), 592–607. `https://doi.org/10.2307/2274701`

The forcing construction used to establish conservativity of **SCOT** and **SCOTS** over **ZF** + **ADC** is simple.

Definition. Let $\mathbb{P} = \{p \subseteq \mathbb{N} \mid p \text{ is infinite}\}$. For $p, p' \in \mathbb{P}$ we say that p' *extends* p (notation: $p' \leq p$) iff $p' \subseteq p$.

Forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{B} = \mathcal{P}^{\infty}(\mathbb{N})/_{\text{fin}}$.

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ be a countable model of **ZF** + **ADC**.
If $\mathcal{G}$ is a generic filter over $\mathbb{P}^{\mathcal{M}}$, the generic extension $\mathcal{M}[\mathcal{G}]$ is a model of **ZF** + **ADC** and the forcing does not add any new reals or countble subsets of M, ie, every countable subset of M in $\mathcal{M}[\mathcal{G}]$ belongs to M.

Working inside $\mathcal{M}[\mathcal{G}]$, the generic filter $\mathcal{G}$ is a nonprincipal ultrafilter over $\mathbb{N}$ and one can construct the ultrapower $(M^{\mathbb{N}}/\mathcal{G}, \in^*)$ of $\mathcal{M}$ by $\mathcal{G}$.

Łoś's Theorem holds in $(M^{\mathbb{N}}/\mathcal{G}, \in^*)$ because **ACC** is available in $\mathcal{M}$, and $\mathcal{M}$ canonically embeds into $(M^{\mathbb{N}}/\mathcal{G}, \in^*)$.

This construction extends $\mathcal{M}$ to a model $\mathcal{N} = (M^{\mathbb{N}}/\mathcal{G}, \in^*, M)$ of **SCOT**.

As described earlier, one can iterate the ultrapower any finite number of times and take a direct limit to obtain an interpretation for **SCOTS**.

The forcing construction used to establish conservativity of **SPOT**$^{+}$ over **ZF** is much more complicated because one needs to force both a generic filter $\mathcal{G}$ on $\mathbb{N}$ and the validity of Łoś's Theorem in the corresponding "ultrapower."

Let $\mathbb{Q} = \{q \in \mathbb{V}^{\mathbb{N}} \mid \exists k \in \mathbb{N}\, \forall i \in \mathbb{N}\, (q(i) \subseteq \mathbb{V}^k \wedge q(i) \neq \emptyset)\}$.

The number k is the *rank* of q. We note that $q(i)$ for each $i \in \mathbb{N}$, and q itself, are sets, but $\mathbb{Q}$ is a proper class.

The forcing notion $\mathbb{H}$ is defined as follows: $\mathbb{H} = \mathbb{P} \times \mathbb{Q}$ and $\langle p', q' \rangle \in \mathbb{H}$ *extends* $\langle p, q \rangle \in \mathbb{H}$ iff p' extends p, $\operatorname{rank} q' = k' \geq k = \operatorname{rank} q$, and for almost all $i \in p'$ and all $\langle x_0, \dots, x_{k'-1} \rangle \in q'(i)$, $\langle x_0, \dots, x_{k-1} \rangle \in q(i)$.

The following proposition establishes a relationship between this forcing and ultrapowers.

"Łoś's Theorem"

Let $\phi(v_1, \ldots, v_s)$ be an $\in$-formula with parameters from $\mathbb{V}$. Then $\langle p, q \rangle \Vdash \phi(\dot{G}_{n_1}, \ldots, \dot{G}_{n_s})$ iff $\operatorname{rank} q = k > n_1, \ldots, n_s$ and $\forall^{\mathbf{aa}} i \in p \, \forall \langle x_0, \ldots, x_{k-1} \rangle \in q(i) \; \phi(x_{n_1}, \ldots, x_{n_s})$.

Iteration of this construction is difficult and conservativity of **SPOTS** over **ZF** is an open problem.