# On the significance of parameters in the comprehension and choice schemata in second-order arithmetic

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NY Set Theory Seminar
December 2022

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*Parameters* are free variables in various axiom schemata in **PA**, **ZFC**, and other similar theories. Given an axiom schema **S**, we let $\mathbf{S}^*$ be the **parameter-free sub-schema**.

Kreisel (*A survey of proof theory, JSL 1968*) was one of those who paid attention to the comparison of some schemata in second-order **PA** and their parameter-free versions. In particular, Kreisel noted that

> [...] if one is convinced of the significance of something like a given axiom schema, it is natural to study details, such as the effect of parameters.

This talk is devoted to the effect of parameters in the schemata of Comprehension and Choice in second-order arithmetic.

The second order Peano arithmetic $\mathbf{PA}_2$ is a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables – for natural numbers $j, k, m, n$ and for sets of them $x, y, z$. The axioms are as follows:

1. Peano's axioms for numbers.
2. **Induction** $\Phi(0) \wedge \forall k \left(\Phi(k) \Longrightarrow \Phi(k+1)\right) \Longrightarrow \forall k \, \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\mathbf{PA}_2)$, and we allow parameters in $\Phi(k)$, *i. e.*, free variables other than $k$.
3. **Extensionality** for sets.
4. **Comprehension** **CA**: $\exists x \, \forall k \left(k \in x \Longleftrightarrow \Phi(k)\right)$, for every formula $\Phi$ in which the variable $x$ does not occur, and we allow parameters in $\Phi$.

$\mathbf{CA}^*$ is the parameter-free sub-schema of **CA** (that is, $\Phi(k)$ contains no free variables other than $k$).

$\mathbf{PA}_2^*$ is the subsystem of $\mathbf{PA}_2$ with **CA** replaced by $\mathbf{CA}^*$.

Is $\mathbf{PA}_2^*$ strictly weaker than $\mathbf{PA}_2$? **Depends.**

In the sense of **consistency**, the answer is NO.
Harvey Friedman (near 1980) established that the theories
$\mathbf{PA}_2$ and $\mathbf{PA}_2^*$ are equiconsistent.

We study the problem in the context of **deductive strength**, and we obtain the opposite answer.

## Theorem 1

$\mathbf{PA}_2^*$ *does not prove that every set has its complement.*

## Proof (sketch)

Let $\mathbf{L}$ be the constructible universe.

Consider a generic extension $\mathbf{L}[\vec{a}]$ obtained by adjoining a Cohen generic sequence $\vec{a} = \langle a_n \rangle_{n<\omega}$ of sets $a_n \subseteq \omega$ to $\mathbf{L}$.

Let $X = (\mathscr{P}(\omega) \cap \mathbf{L}) \cup \{a_n : n < \omega\}$.

Then $X$ is an $\omega$-model of $\mathbf{PA}_2^*$ in which sets $a_n$ do not have their complements, and hence the full $\mathbf{CA}$ fails.

The proof of $\mathbf{CA}^*$ in $X$ is based on the standard homogeneity and permutation-related properties of the forcing notion $\mathbb{C}^\omega$ involved, which is the finite-support product of the Cohen forcing $\mathbb{C}$. □

Thus $\mathbf{PA}_2^*$ does not prove a simple consequence of the full $\mathbf{CA}$.

**Theorem 2**

$\mathbf{PA}_2^* + \mathbf{CA}(\boldsymbol{\Sigma}_2^1)$ *does not prove a certain instance of the full* $\mathbf{CA}$.

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# Theorem 2 proof sketch, basic model

TOC

Let $\mathbf{L}$ be the constructible universe.

Using a version of the product/iterated Sacks forcing, we define:

- the index set $\mathbb{I} = (\omega_1)^{<\omega} \smallsetminus \{\Lambda\} \in \mathbf{L}$ — $\omega_1$ is used for $\omega_1^{\mathbf{L}}$;
- reals $b_s \subseteq \omega$, $s \in \mathbb{I}$, such that each $b_{s^\frown\alpha}$ is Sacks generic over $\mathbf{L}[b_s]$, and in addition we put $b_\Lambda = \varnothing$ for the empty tuple $\Lambda$;
- the whole array $\vec{b} = \langle b_s \rangle_{s \in \mathbb{I}}$ of those reals.
- the generic extension $\mathbf{L}[\vec{b}]$, which is the basic model.

This is a kind of generalized iteration of the Sacks forcing, along the index set $\mathbb{I}$; we may call it an arboreal iteration.

The version $\mathbb{P}$, of the product/iterated Sacks forcing we use, consists of all forcing conditions defined in **L** as follows.

**A** Let $\xi \subseteq \mathbb{I}$ be any countable initial segment of the index set $\mathbb{I} = (\omega_1)^{<\omega} \smallsetminus \{\Lambda\}$.

**B** We consider $\mathscr{P}(\omega)$ as identic to $2^\omega$, so that both $\mathscr{P}(\omega)$ and $\mathscr{P}(\omega)^\xi$ are Polich compact spaces;

**C** Let $H : \mathscr{P}(\omega)^\xi \to \mathscr{P}(\omega)^\xi$ be a homeomorphism, projection-keeping in the sense that if $\eta \subseteq \xi$ is an initial segment and $x, y \in \mathscr{P}(\omega)^\xi$ then $x \restriction \eta = y \restriction \eta \iff H(x) \restriction \eta = H(y) \restriction \eta$.

Consider the (closed) set $X_H = \mathtt{ran}\, H = \{H(x) : x \in \mathscr{P}(\omega)^\xi\}$.

**D** The forcing $\mathbb{P}$ consists of all such sets $X_H$; put $\mathbf{dim}\, X_H = \xi$.

**E** Put $X \leq Y$ ($X$ is stronger) iff $\eta = \mathbf{dim}\, Y \subseteq \mathbf{dim}\, X$ and $X \restriction \eta \subseteq Y$.

This is a kind of generalized iteration of the Sacks forcing.

Let **L** be the constructible universe.

Using the forcing notion $\mathbb{P}$, we define:

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- the whole $\mathbb{P}$-generic array $\vec{b} = \langle b_s \rangle_{s\in\mathbb{I}}$ of those reals.
- the $\mathbb{P}$-generic extension $\mathbf{L}[\vec{b}]$, which is the basic model.

This is a kind of generalized iteration of the Sacks forcing, along the index set $\mathbb{I}$; we may call it an arboreal iteration.

**Lemma (strict successor lemma)**

*Let* $s \in \mathbb{I} \cup \{\Lambda\}$.

i. *If* $\gamma < \omega_1$ *then* $b_{s^\frown\gamma}$ *is a* **strict successor** *of* $b_s$ *in the sense that* $b_s <_{\mathbf{L}} b_{s^\frown\gamma}$ *and if* $x \subseteq \omega$, $x <_{\mathbf{L}} b_{s^\frown\gamma}$ *then* $x \leq_{\mathbf{L}} b_s$.

ii. *If* $y \in \mathbf{L}[\vec{b}]$, $y \subseteq \omega$ *is a strict successor of* $b_s$ *then there is an ordinal* $\gamma < \omega_1$ *such that* $y \equiv_{\mathbf{L}} b_{s^\frown\gamma}$.

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Consider the set

$$W = W(\vec{b}) = \{b_\gamma : \gamma < \omega_1\} \cup \{b_{\gamma^\frown 0^n} : \gamma < \omega_1 \wedge n < \omega\} \cup \\ \cup \{b_{\gamma^\frown 0^n {}^\frown 1} : \gamma < \omega_1 \wedge n \in b_{\gamma^\frown 1}\}.$$

Thus $W \in \mathbf{L}[\vec{b}]$.

Let $X = \left(\bigcup_{Z \subseteq W \text{ finite}} \mathbf{L}[Z]\right) \cap \mathscr{P}(\omega)$.

**Then $X$ proves** Theorem 2**.**

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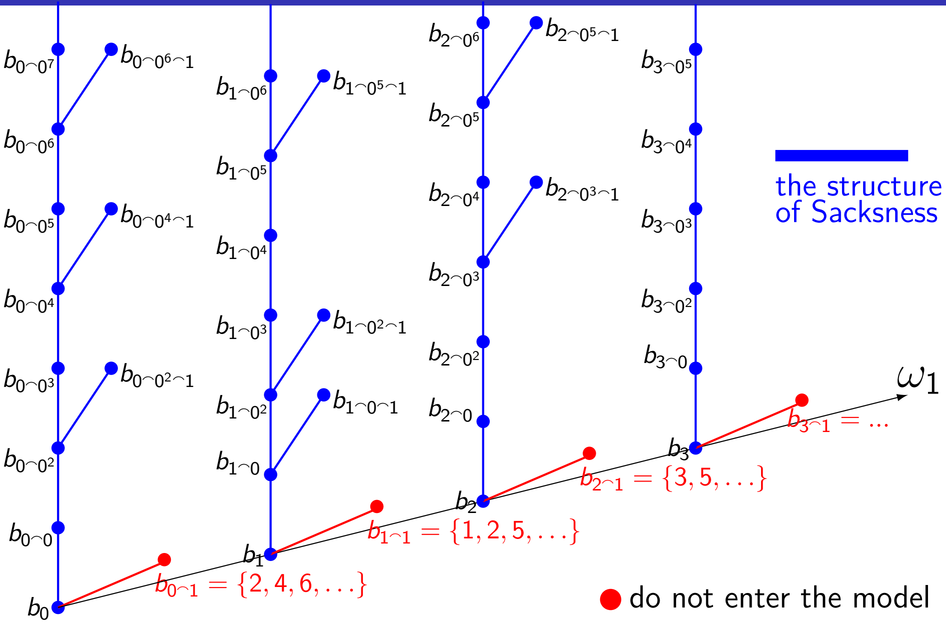
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**Then $X$ proves Theorem 2.**

- $X$ is a model of $\mathbf{PA}_2^*$ by the permutation technique,
- $X$ is a model of $\mathbf{CA}(\boldsymbol{\Sigma}_2^1)$ by the Shoenfield absoluteness,
- the sets $b_{\gamma^\frown 1}$ do not belong to $X$ by construction,
- yet each $b_{\gamma^\frown 1}$ is definable in $X$, with $b_\gamma$ as the only parameter, by means of the structure of the Sacksness above $b_\gamma$ and the Lemma on strict succesors — thus **CA fails** in $X$.

  More specifically, $\mathbf{CA}(\Sigma_4^1)$ fails.

Prove that $\mathbf{PA}_2$ is not finitely axiomatizable over $\mathbf{PA}_2^*$.

More specifically, prove that, for any $n \geq 2$,
$\mathbf{PA}_2^* + \mathbf{CA}(\boldsymbol{\Sigma}_n^1)$ does not imply an instance of $\mathbf{CA}^*(\Sigma_{n+1}^1)$.

Case $n = 2$ is **partially** established by the proof of Theorem 2 above, as the counterexample is unfortunately more complicated than $\Sigma_3^1$ as of yet.

**Work in progress.**

**Choice, $\mathbf{AC}_\omega$:** $\forall k\, \exists x\, \Phi(k,x) \Longrightarrow \exists x\, \forall k\, \Phi(k,(x)_k)$,
where $(x)_k = \{m : 2^m(2k+1)-1 \in x\}$.

**Dependent Choice, DC:** $\forall x\, \exists y\, \Phi(x,y) \Longrightarrow \exists x\, \forall k\, \Phi((x)_k,(x)_{k+1})$.

Let $\mathbf{AC}^*_\omega$ be the parameter-free sub-schema. ($\mathbf{DC} \Longleftrightarrow \mathbf{DC}^*$ is known.)

**The Levy model for $\mathbf{PA}_2 + \neg\mathbf{AC}^*_\omega$:** extend $\mathbf{L}$ by the Levy-collapse below $\aleph_\omega$, so that $\aleph_1 = \aleph_\omega^{\mathbf{L}}$ holds in the extension.

**Guzicki's model for $\mathbf{PA}_2 + \mathbf{AC}^*_\omega + \neg\mathbf{AC}_\omega$:** extend $\mathbf{L}$ by the Levy-collapse below $\aleph_{\omega_1}$, so that $\aleph_1 = \aleph_{\omega_1}^{\mathbf{L}}$ holds in the extension. (Any real that codes a collapse of $\aleph_1^{\mathbf{L}}$ can serve as a parameter for the violation of $\mathbf{AC}_\omega$.)

**A common shortcoming of the two models:** the necessary use of cardinals out of the scope of $\mathbf{PA}_2$ in the collapse forcing method.

**We'll show how to fix this problem.** We'll present non-collapse models for $\mathbf{PA}_2 + \neg\mathbf{AC}^*_\omega$, for $\mathbf{PA}_2 + \mathbf{AC}^*_\omega + \neg\mathbf{AC}_\omega$, and for $\mathbf{PA}_2 + \mathbf{AC}_\omega + \neg\mathbf{DC}$.

## A non-collapse model for $\neg \mathbf{AC}^*_\omega$ TOC

We start with the same arboreal Sacks generic extension $\mathbf{L}[\vec{b}]$ (the basic model), where $\vec{b} = \langle b_s \rangle_{s \in \mathbb{I}}$ is a $\mathbb{P}$-generic array of reals $b_s \subseteq \omega$, such that each $b_{s^\frown\alpha}$ is Sacks generic over $\mathbf{L}[b_s]$.

Recall that $\mathbb{I} = (\omega_1)^{<\omega} \smallsetminus \{\Lambda\} \in \mathbf{L}$ is the index set.

We let $\Omega \in \mathbf{L}$ be the set of all countable or finite initial segments $\xi \subseteq \mathbb{I}$ such that there is a number $n = n_\xi < \omega$ with $\operatorname{dom} s \le n$ for all $s \in \xi$.

Let $W = W(\Omega, \vec{b}) = \{\vec{b} \restriction \xi : \xi \in \Omega\}$; then $W \in \mathbf{L}[\vec{b}]$.

**Theorem** (a non-collapse model for $\mathbf{PA}_2 + \neg \mathbf{AC}^*_\omega$)

*Under the assumptions above, the class $\mathfrak{M} = \mathbf{HOD}(W)$, of all sets hereditarily $W$-ordinal-definable in $\mathbf{L}[\vec{b}]$ (parameters are elements of $W$), is a model of $\mathbf{ZF}$ in which the countable parameter-free $\mathbf{AC}^*_\omega$ fails. Accordingly, if $X = \mathscr{P}(\omega) \cap \mathfrak{M}$, then $X$ is a model of $\mathbf{PA}_2 + \neg \mathbf{AC}^*_\omega$.*

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# Two more non-collapse models

We work with the same arboreal Sacks generic extension $\mathbf{L}[\vec{b}]$.

Let $\Omega' \in \mathbf{L}$ be the set of all ctble initial segments $\xi \subseteq \mathbb{I}$ such that for any $\alpha < \omega_1$ there is $n < \omega$ satisfying $\mathtt{dom}\, s \leq n$ for all $s \in \xi$ with $s(0) = \alpha$.

Let $\Omega'' \in \mathbf{L}$ be the set of all countable well-founded initial segments $\xi \subseteq \mathbb{I}$.

We put $W' = \{\vec{b} \restriction \xi : \xi \in \Omega'\}$ and $W'' = \{\vec{b} \restriction \xi : \xi \in \Omega''\}$.

**Theorem** (two non-collapse models)

*Under the assumptions above, it is true in* $\mathbf{L}[\vec{b}]$ *that:*

i *the class* $\mathfrak{M}' = \mathbf{HOD}(W')$ *is a model of* **ZF** *in which the ctble param.-free* $\mathbf{AC}^*_\omega$ *holds but the ctble* $\mathbf{AC}_\omega$ *with parameters fails.*

ii *the class* $\mathfrak{M}'' = \mathbf{HOD}(W'')$ *is a model of* **ZF** *in which the countable* $\mathbf{AC}_\omega$ *holds but* **DC** *fails.* — *Jensen's old technique.*

*Accordingly,* $\mathscr{P}(\omega) \cap \mathfrak{M}'$ *is a model of* $\mathbf{PA}_2 + \mathbf{AC}^*_\omega + \neg\mathbf{AC}_\omega$, *whereas* $\mathscr{P}(\omega) \cap \mathfrak{M}''$ *is a model of* $\mathbf{PA}_2 + \mathbf{AC}_\omega + \neg\mathbf{DC}$.

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Non-cardinal-collapse models are presented for the following theories:

1. $\mathbf{PA}_2^* + \neg\mathbf{CA}$ — by means of the Cohen forcing
2. $\mathbf{PA}_2^* + \mathbf{CA}(\mathbf{\Sigma}^1_2) + \neg\mathbf{CA}$ — this and **ff** by arboreal Sacks iterations, the counter-example estimated to be $\mathbf{CA}(\mathbf{\Sigma}^1_4)$;
3. $\mathbf{PA}_2 + \neg\mathbf{AC}_\omega^*$ — with a $\mathit{\Pi}^1_3$ counter-example;
4. $\mathbf{PA}_2 + \mathbf{AC}_\omega^* + \neg\mathbf{AC}_\omega$ — with a $\mathbf{\Pi}^1_3$ counter-example; ;
5. $\mathbf{PA}_2 + \mathbf{AC}_\omega + \neg\mathbf{DC}$ — with a $\mathit{\Pi}^1_3$ counter-example. — Gitman-SDF-K, 2019, JML — with a $\mathit{\Pi}^1_2$ counterexample but using a way more complicated technique of iterated Jensen-minimal forcing.

All ensuing consistency results do not involve cardinal collapse and are manageable on the base of $\mathbf{ZFC}^-$ (*sans* the Power Set axiom), hence in principle **on the base of $\mathbf{PA}_2$ itself**.

The speaker thanks **the organizers** for the opportunity to give this talk

The speaker thanks **everybody** for interest and patience