Bounded finite set theory

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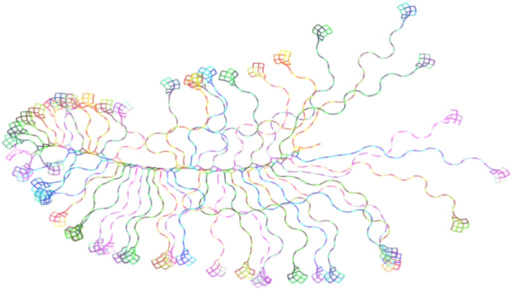


Arithmetic and finite set theory

The correspondence — does it work for bounded arithmetic?

$$\mathsf{FST} : \mathsf{PA} \quad = \quad ? : I\Delta_0$$

$$\begin{aligned} \mathsf{FST} &= \text{Finite Set Theory} \\ &= \mathsf{ZF} - \mathsf{Inf} + \neg\mathsf{Inf} \; (+ \mathsf{TC}) \end{aligned}$$

$$\mathsf{TC} = \text{Axiom of Transitive Containment}$$

- FST and PA are mutually interpretable. Small print
- Any model of PA is isomorphic to the arithmetic of the ordinals of a model of FST.

Arithmetic and finite set theory

The correspondence via Ackermann's interpretation

Let $x \in_{Ack} y$ be the predicate expressing that the coefficient of 2^x in the binary expansion of y is 1. Then

- $\langle \mathbb{N}, \in_{Ack} \rangle \cong \langle V_\omega, \in \rangle$.
- If $M \models \mathsf{PA}$, then $Ack_M =_{df} \langle M, \in^M_{Ack} \rangle \models \mathsf{FST}$ and its ordinals are isomorphic to M.
- Corollary: PA interprets FST.

Arithmetic and finite set theory

The correspondence via induction

- Adjunction: $x; y = x \cup \{y\}$
- Work in the language $\mathcal{L}(0;)$
- $\in$ is definable: $y \in x \leftrightarrow x; y = x$
- PS_0 consists of:

$$0; x \neq 0$$

$$[x; y]; z = [x; z]; y$$

$$[x; y]; z = x; y \leftrightarrow x; z = x \vee z = y$$

Arithmetic and finite set theory

The correspondence via induction

Tarski-Givant induction:

$$\varphi(0) \ \wedge\ \forall x \forall y(\varphi(x) \wedge \varphi(y) \rightarrow \varphi(x;y)) \ \rightarrow\ \forall x \varphi(x).$$

PS consists of PS_0 together with induction for each first order φ (with parameters). (Previale)

- PS is logically equivalent to $\mathsf{ZF} - \mathsf{Inf} + \neg\mathsf{Inf} + \mathsf{TC}$
- We're "arithmetizing" set theory in the sense of basing it on an induction principle over a successor operator.

$I\Sigma_1 S$

is enough to Ackermannize

$I\Sigma_1 S$ has induction for Σ_1 formulæ in the Lévy hierarchy.

$$\mathsf{PS} : \mathsf{PA} \quad = \quad I\Sigma_1 S : I\Sigma_1$$

- If $M \models I\Sigma_1$, then $Ack_M \models I\Sigma_1 S$ and the ordinals of Ack_M, together with the restrictions of addition and multiplication to them, are isomorphic to M.
- Parsons' Theorem transfers to set theory: the primitive recursive set functions are those provably total in $I\Sigma_1 S$, where...

The primitive recursive set functions

are obtained from the initial functions

- the constant function $\tilde{0}(\vec{x}) = 0$,
- projections, and
- adjunction $x; y$,

by closing under

- *substitutions* $f(\vec{x}) = g(h_1(\vec{x}), \cdots, h_k(\vec{x}))$
- and *recursion* of form

$$f(0, \vec{z}) = g(\vec{z})$$
$$f([a; p], \vec{z}) = h(a, p, f(a, \vec{z}), f(p, \vec{z}), \vec{z})$$

$\mathcal{L}(0\,;<)$

"Bounded with respect to what?" — a transitive relation is needed …

…so we add $<$ to our language, intended to mean the transitive closure of the $\in$ relation.
Let $\mathsf{PS}_0^<$ be the result of adding to PS_0:

$$x \not< 0 \quad \text{and} \quad x < y;z \leftrightarrow x < y \lor x \leq z$$

Then we define the class of Δ_0 formulæ in the expanded language by allowing bounded quantification of form $\forall y < t$, $\exists y < t$ where t is a term. And we define $I\Delta_0 S$ to be $\mathsf{PS}_0^<$ together with induction for Δ_0 formulæ in the expanded language.

- In $I\Sigma_1 S$ this doesn't matter because we have the transitive closure so $<$ is definable in $\mathcal{L}(0\,;)$.

The primitive recursive set functions

include set-theoretic operators such as P, $\cup$, $\bigcup$, $|x| =$ cardinality of x, $\mathsf{TC}(x) =$ transitive closure of x, V_n, and ordinal arithmetic operations $+, \cdot, x^y$.

$I\Delta_0 S(\cup)$ means: $I\Delta_0 S$ plus "$\cup$ is total".
Or equivalently: $I\Delta_0 S$ in language expanded by a function symbol $\cup$ and axioms:

$$x \cup 0 = x \quad \text{and} \quad x \cup [y; z] = (x \cup y); z$$

and similarly for other primitive recursive functions.

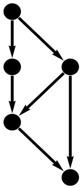
Sets as digraphs

(Aczel)

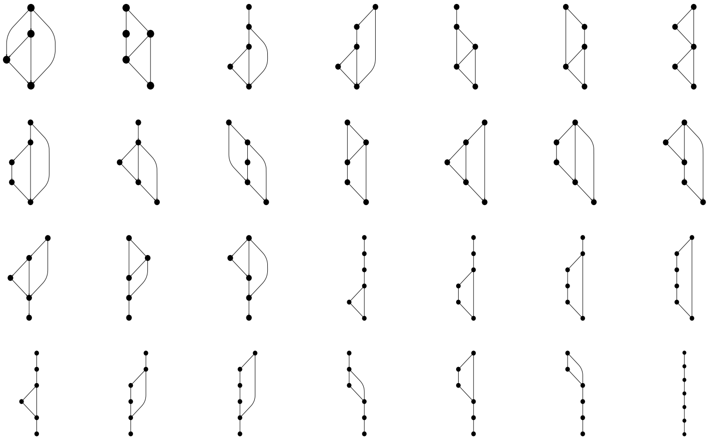
Each HF set x is uniquely specified by a finite extensional acyclic digraph with a single source

$$G(x) = \{\langle y, z \rangle \mid z \in y \leq x\}$$

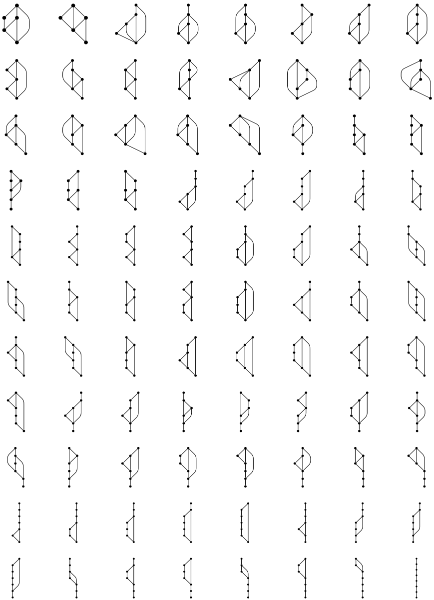
e.g. $c = \{\{\{0\}\}, \{0, \{0\}\}\} =$ the "pair of deuces"



The 28 sets whose graphs have 6 edges



The 88 sets whose graphs have 7 edges



The ordinals of a model of $I\Delta_0 S$

Interpreting arithmetic in set theory

- Given $V \models I\Delta_0 S$, we want to talk about the ordinal arithmetic of V.

Von Neumann ordinals (1923) (Zermelo, Mirimanoff): $n+1 = n; n = n \cup \{n\}$

Zermelo ordinals (1908): $(n+1)_z = 0; n_z = \{n_z\}$

We shall see that they can differ in a model of $I\Delta_0 S$.

Zermelo ordinals are simpler

in setbuilder notation

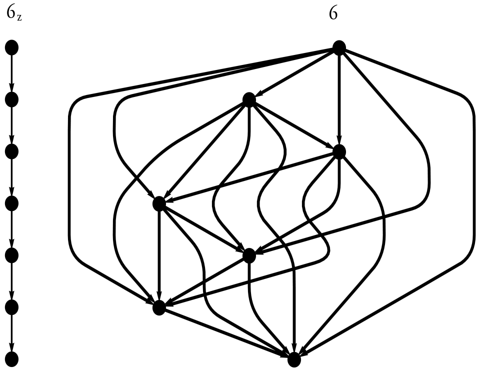
Zermelo: $6_z = \{\{\{\{\{\{\}\}\}\}\}\}$

Von Neumann:

$$
\begin{aligned}
6 =&\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \\
&\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\}, \\
&\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \\
&\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}\}
\end{aligned}
$$

Zermelo ordinals are simpler

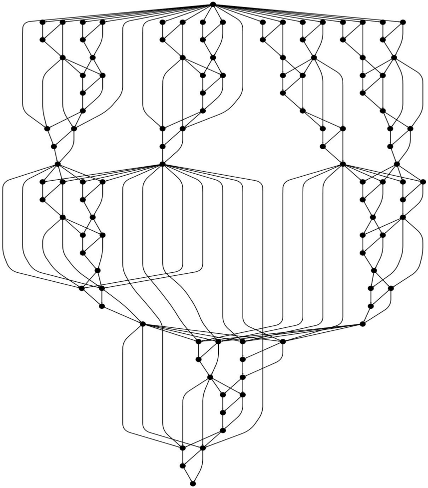
as digraphs



This time, only polynomially so.

c^c

in the Zermelo arithmetic where c is the "pair of deuces"



PS : PA $= I\Delta_0 S : I\Delta_0$?

Proposition. Suppose $V \models I\Delta_0 S$ and W is a transitive subset of V closed under adjunction. Then Δ_0 formulæ are absolute between V and W, and $W \models I\Delta_0 S$.

- Q1: Which axioms of set theory are provable in $I\Delta_0 S$?
- Q2: Given $M \models I\Delta_0$, is there a model of $I\Delta_0 S$ whose ordinal arithmetic is isomorphic to M?

Which axioms of ZF are provable in $I\Delta_0 S$?

- $I\Delta_0 S \vdash$ the Pair Set Axiom, Extensionality, $\neg\mathsf{Inf}$, and the Axiom of Foundation.
- $I\Delta_0 S(\mathsf{TC}, \mathsf{P}) \vdash \bigcup$, i.e. the Union Axiom. This is because $\bigcup x \in \mathsf{P}(\mathsf{TC}(x))$.
- $I\Delta_0 S(\mathsf{P}) \vdash \Delta_0$-Comprehension.
- Does $I\Delta_0 S \vdash \Delta_0$-Comprehension? ... If so, and if the answer to Q2 is positive, then $I\Delta_0 \vdash \Delta_0 PHP$. This is because $I\Delta_0 S$ proves a pigeon hole principle for functions which are sets.

Submodels of Ack_M

for $M \models I\Sigma_1$

- For $I \subseteq_e M$: $V_I = \bigcup_{i \in I} V_i$.
- $V_I \models I\Delta_0 S(\bigcup, \mathsf{TC}, \mathsf{P})$.

- H_i is the set of all elements of $V_M = Ack_M$ whose transitive closure has cardinality $< i$, i.e. all sets of hereditary cardinality $< i$, i.e all sets whose digraph representations have $\leq i$ nodes.
- If I is closed under $+$, then $H_I \models I\Delta_0 S(\bigcup, \mathsf{TC})$.
- $H_I \models \mathsf{P}$ iff I is closed under exponentiation.

Submodels of Ack_M

for $M \models I\Sigma_1$

- $C_i = \{x \in V_M \mid V_M \models \forall y \leq x \, |y| < i\}$.
- Let $e_0 = 1$, $e_{n+1} = 2^{e_n}$.
- *Theorem:*
  (1) $V_I \cap C_J \models I\Delta_0 S$.
  (2) $V_I \cap C_J \models \cup$ iff $J \geq e_I$ or J is closed under addition.
  (3) $V_I \cap C_J \models \bigcup$ iff $J \geq e_I$ or J is closed under multiplication.

  (5) $V_I \cap C_J \models \mathsf{P}$ iff $J \geq e_I$ or J is closed under exponentiation.

Submodels of Ack_M

and independence results

- (4)(i) Suppose I is closed under addition. Then $V_I \cap C_J \models \mathsf{TC}$ iff $J \geq e_I$ or $J^I = J$.

  (4)(ii) $V_I \cap C_J \models \mathsf{TC}$ iff $J \geq e_I$ or $\exists i \in I(J^{I-i} = J \wedge e_i \in J)$.
- This theorem provides examples to show that e.g. $I\Delta_0 S \nvdash \cup$ and $I\Delta_0 S(\bigcup, \mathsf{P}) \nvdash \mathsf{TC}$.
- Does $I\Delta_0 S(\mathsf{TC}) \vdash \bigcup$?
- In $V_I \cap C_J$ with $J < I$, the von Neumann ordinals are J but the Zermelo ordinals are I.

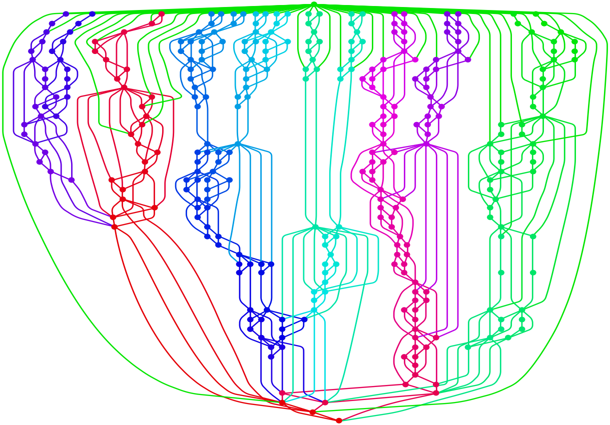
Ordinals in the Ackermann interpretation

Q2: Given $M \models I\Delta_0$, is there a model of $I\Delta_0 S$ whose ordinal arithmetic is isomorphic to M?

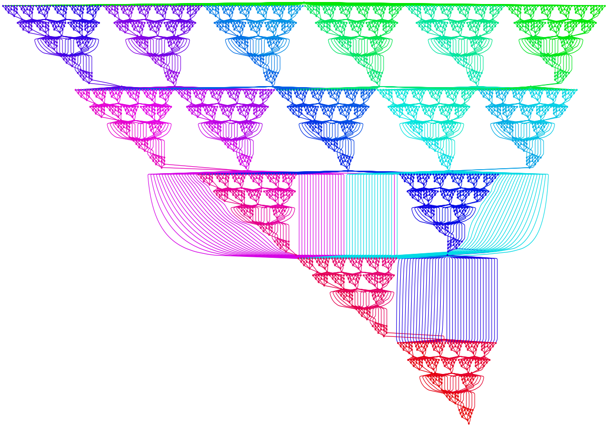
The Ackermann code for n_z is e_{n-1}. The Ackermann code for the von Neumann ordinal n is even bigger. This iterated-exponential growth means that:

- The Ackermann interpretation gives:
  Theorem: $I\Delta_0 + \mathsf{Exp}$ interprets $I\Delta_0 S$.
- But the Ackermann interpretation fails to preserve ordinals if M is not a model of "$n \mapsto e_n$ is total".

Generating set digraphs



Generating set digraphs

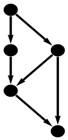


Models of $I\Delta_0 + \mathsf{Exp}$ are expandable

Q2: Given $M \models I\Delta_0$, is there a model of $I\Delta_0 S$ whose ordinal arithmetic is isomorphic to M?

Yes if M has an end extension to a model of $I\Sigma_1$.

Theorem: Yes if $M \models \mathsf{Exp}$.
Idea: Code sets by their digraph representations, e.g.



$c = \{\{\{0\}\}, \{0, \{0\}\}\}$ = the "pair of deuces" is represented by $s^* = \langle\{0\}, \{1\}, \{0, 1\}, \{2, 3\}\rangle$ which is represented in turn by $s = \langle 1, 2, 3, 12\rangle$.

Models of $I\Delta_0 + \mathsf{Exp}$ are expandable

Definition: A *σ-sequence* in M is a strictly increasing sequence $s = \langle s_1, \ldots, s_n \rangle$ such that for each i, $0 < s_i < 2^i$.

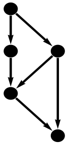
If s is a σ-sequence, define $s_i^* = \{j < i \mid j \in_{Ack} s_i\}$ and s^* to be the corresponding sequence $\langle s_1^*, \ldots, s_n^* \rangle$. (Peddicord)

Then $s_i^* \subseteq \{0, \ldots, i-1\}$ and the s_i^* are distinct and non-empty.

The idea is to use the sequence s to represent the set whose digraph has nodes $0, \ldots, n$ with an edge from j to i just when $i \in s_j^*$.

Generating set digraphs

The digraph interpretation



is represented by $s^* = \langle \{0\}, \{1\}, \{0,1\}, \{2,3\} \rangle$,
$s = \langle 1, 2, 3, 12 \rangle$.
But also by $t^* = \langle \{0\}, \{0,1\}, \{1\}, \{2,3\} \rangle$,
$t = \langle 1, 3, 2, 12 \rangle$ ... but this is not increasing!

Generating set digraphs

The digraph interpretation

We also need a condition on a σ-sequence to ensure the corresponding digraph only has one source:

Definition: $s = \langle s_1, \ldots, s_n \rangle$ is *lean* iff $\forall i < n \; \exists j \leq n \; i \in_{Ack} s_j$.

Fact: Every HF set is represented by a *unique* lean σ-sequence.

Models of $I\Delta_0 + \mathsf{Exp}$ are expandable

The digraph interpretation

Given $M \models I\Delta_0 + \mathsf{Exp}$ let D^M = the set of lean σ-sequences in M. Adjunction in D^M is defined as a binary operation on σ-sequences that mimics the surgery on digraphs needed to form an adjunction of the corresponding sets. The relation $<$ is interpreted similarly.

Theorem: Let $M \models I\Delta_0 + \mathsf{Exp}$. Then $D^M \models I\Delta_0 S$ and the (von Neumann or Zermelo) ordinals of D^M are isomorphic to M.

Because the construction of D^M in M is Δ_0-defined and uniform, this gives an interpretation of $I\Delta_0 S$ in $I\Delta_0 + \mathsf{Exp}$.

Interpretability: Summary

- $I\Delta_0 S(+,\cdot)$ interprets $I\Delta_0$ in two ways (von Neumann and Zermelo) which are not necessarily equivalent.
- $I\Delta_0 + \mathsf{Exp}$ interprets $I\Delta_0 S$ in two ways. But only the digraph interpretation preserves ordinals.
- Does $I\Delta_0$ interpret $I\Delta_0 S$?