Model theoretic characterizations of truth
Part I
(joint work with Bartosz Wcisło)

Mateusz Łełyk

Faculty of Philosophy, University of Warsaw

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Introduction

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Most of the time we shall work with $B = \mathrm{PA}$, but it can be clearly seen where induction is not needed.

Two main characters

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$\mathrm{UTB}^-[B]$ extends B with all sentences of the form

$$\forall x \big(T(\ulcorner \phi(\dot{x}) \urcorner) \equiv \phi(x) \big),$$

for $\phi(x) \in \mathcal{L}_B$. UTB denotes $\mathrm{UTB}^-[\mathrm{PA}] + \mathrm{Ind}(\mathcal{L}_T)$.

Properties of Tarski biconditionals

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2. $\exists T \subseteq \mathcal{M}, (\mathcal{M}, T) \models \mathrm{UTB}$.

Moreover the implication $(2) \Rightarrow (1)$ *holds in arbitrary model.*

A prototypical result

Proposition (Kossak)

Suppose $U \supseteq \mathsf{PA} + \mathsf{Ind}_{\mathcal{L}_U}$ *is a theory in a countable language, such that for every* $\mathcal{M} \models U$, $\mathcal{M}\restriction_{\mathcal{L}_{\mathsf{PA}}}$ *is (short) recursively saturated. Then for every* $\mathcal{M} \models U$ *there exists* $T \in Def(\mathcal{M})$ *such that* $(\mathcal{M}, T) \models \mathsf{UTB}$.

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Definition

Let U, V be any first-order theories in purely relational signatures. We say that U *semantically defines* V if for every model $\mathcal{M} \models U$ and every $R \in \mathcal{L}_V$ there exists $A_R \in \mathit{Def}(\mathcal{M})$ such that $(\mathcal{M}, \{A_R\}_{R \in \mathcal{L}_V}) \models V$.

What's in logician's cuffs?

Proposition

Suppose that $(\mathcal{M}, T)$, $(\mathcal{N}, T')$ *are two models of* $\mathrm{UTB}^-[\mathrm{EA}]$. *Then*

$$(\mathcal{M}, T) \subseteq (\mathcal{N}, T') \Longrightarrow \mathcal{M} \preceq \mathcal{N}.$$

$\phi \in \Sigma_n^*$ iff ϕ starts with at most n alternating blocks of quantifiers (starting with $\exists$) followed by an *atomic* formula.

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Definition

Let U extend EA. We say that U uniformly imposes $\mathcal{L}$-elementarity if there is an $n \in \omega$ such that for every $\mathcal{M}, \mathcal{N} \models U$

$$\mathcal{M} \preceq_{\Sigma_n^*} \mathcal{N} \Longrightarrow \mathcal{M}\restriction_{\mathcal{L}} \preceq \mathcal{N}\restriction_{\mathcal{L}}.$$

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Question (Kossak's "off the cuff" question, formal version)

Does every reasonable theory (extending EA*) which uniformly imposes arithmetical elementarity, semantically define* $\mathrm{UTB}^-[\mathrm{EA}]$*?*

Variants of the main problem

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We say that U imposes $\mathcal{L}$-elementarity (elementary equivalence) if for every $\mathcal{M}$ there is an $n \in \omega$ such that for every $\mathcal{N} \models U$

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We say that U (syntactically) defines V if for every $R \in \mathcal{L}_V$ there is a formula $\phi_R(\bar{x}) \in \mathcal{L}_U$ such that for every Φ-axiom of V

$$U \vdash \Phi[\phi_R(\bar{t})/R(\bar{t})]_{R \in \mathcal{L}_V}.$$

Infinite pathologies

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From now on we shall restrict ourselves to r.e. theories in finite languages extending EA. By pairing, we can safely assume that such theories has just one additional predicate P. For simplicity, we specialize to the case of theories extending EA and talk only about imposing arithmetical elementarity (elementary equivalence).

For starters: theories with full induction

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Suppose that $U \vdash \mathrm{Ind}_{\mathcal{L}_U}$ *and* U *uniformly imposes elementarity (elementary equivalence). Then* U *(syntactically) defines* UTB *(resp.* TB*).*

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$$\mathcal{N} \models T(\phi(a)) \iff \mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(a).$$

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For the proofs see the blackboard.

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$$T_i^{\mathcal{M}_j}(x) := (c_{\mathcal{M}_i})^{\mathcal{M}_j}.$$

Then $(\mathcal{M}_0, T_0, T_1) \preceq_n (\mathcal{M}_1, T_0, T_1)$ and they are both models of U.

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$$\forall \psi \in \Sigma_n^* \Big(\mathrm{Sat}(\mathrm{True}_{\Sigma_n}(\psi(\dot{x})), \mathrm{dp}(\ulcorner \phi \urcorner)) \rightarrow \mathrm{True}_{\Sigma_n}(\ulcorner \psi(\phi(\dot{x})) \urcorner) \Big)$$

Thank you for your attention.