Broad Infinity and Generation Principles

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Outline

From Infinity to Broad Infinity

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It expresses a similar intuition: the set of G-broad numbers.

How strong is that intuition? You be the judge!

Standard theories: classical

ZFC is a set theory that includes the Axiom of Choice.

It assumes that everything is a set, and that $\in$ is well-founded.

Variations

- ZFCA allows urelements.
- ZFCN allows non-well-founded sets.
- ZFCAN allows both.

They are all equally good for the purposes of this talk.

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They are all equally good for the purposes of this talk.

ZF, ZFA, ZFN, ZFAN are the same, without the Axiom of Choice.

Standard theories: intuitionistic/constructive

CZF $\subseteq$ IZF $\subseteq$ ZF $\subseteq$ ZFC.

IZF and CZF don't have the law of Excluded Middle (nor AC).

IZF has Truth Value Set (hence Powerset), CZF doesn't.

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Warning The words "intuitionistic", "constructive" and "predicative" are used in various conflicting ways.

Be careful with ordinals, e.g. the following are not provable.

- Any two ordinals are related by $<, >, =$. (Trichotomy)
- Every ordinal is 0, successor or limit.
- Any inhabited set of ordinals has a least element.

Sets and Classes

Set theorists often speak about classes, which can be sets or proper classes such as Ord.

Using the language of ZFC, the only classes we can speak of are given by formulas with parameters.

Likewise for functions on a class C, classes dependent on $x \in C$, etc.

Statements $\forall C$ are expressed as schemes.

Statements $\exists C$ are (usually) constructions.

Note: we allow extra predicate and function symbols.

Blass's Axiom

A regular ordinal α can't be expressed as a supremum of α smaller ordinals.

Blass's Axiom: The class of regular ordinals is unbounded.

Not provable in ZF, assuming the consistency of the existence of arbitrarily large strongly compact cardinals. [Gitik]

Provable in ZFC. So it's a "little bit choicy".

The Ord-is-Mahlo scheme

Ord-is-Mahlo: Every closed unbounded class of ordinals contains a regular ordinal.

Can't be proved in ZFC, assuming ZFC is consistent.

Implies Blass's Axiom. So it's a "little bit choicy".

Broad Infinity vs Ord-is-Mahlo

Broad Infinity and Ord-is-Mahlo are equivalent, assuming AC.

Conjecture: Broad Infinity doesn't imply Blass's Axiom, under some consistency hypothesis.

"Broad Infinity is Ord-is-Mahlo minus the little bit of Choice."

This allows us to separate distinct intuitions: Broad Infinity and AC.

Four Principles of Infinity

1. Axiom of Infinity
2. Signature Infinity
3. Reduced Broad Infinity
4. Broad Infinity

Infinity: my favourite version

$\mathcal{T}$ is the class of all things (universal class).

First step: the constructors

We want Zero $\in \mathcal{T}$ and Succ : $\mathcal{T} \to \mathcal{T}$ such that Succ is injective and never yields Zero.

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We want Zero $\in \mathcal{T}$ and Succ : $\mathcal{T} \to \mathcal{T}$ such that Succ is injective and never yields Zero.

Zermelo's definition achieves this:

$$\begin{aligned} \mathsf{Zero} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Succ}(x) &\stackrel{\text{def}}{=} \{x\} \end{aligned}$$

Infinity (continued)

A set X is nat-inductive when

- Zero $\in X$
- for any $x \in X$, we have Succ$(x) \in X$.

A set of all natural numbers is a minimal (and therefore least) nat-inductive set.

Infinity (continued)

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Axiom of Infinity: There is a set of all natural numbers.

Note that this uniquely specifies a set.

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Example of a natural number

Succ(Succ(Succ(Zero)))

Signature Infinity

A signature $S = (K_i)_{i \in I}$ is a family of sets.

I is a set of symbols, and K_i is the arity of i.

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A set X is S-inductive when

- for any symbol $i \in I$ and K_i-tuple $[a_k]_{k \in K_i}$ within X, we have $\langle i, [a_k]_{k \in K_i} \rangle \in X$.

A set of all S-terms is a minimal (and therefore least) S-inductive set.

Theorem (IZF): Signature Infinity

For every signature S, there's a set of all S-terms.

Example of an S-term

Symbol	Arity
5	$\{0,1,2,3\}$
6, 7	$\emptyset$
8	$\{0,1,2\}$

$$\left\langle 8, \begin{bmatrix} 0 & \mapsto & \left\langle 5, \begin{bmatrix} 0 & \mapsto & \langle 7, [\,]\rangle \\ 1 & \mapsto & \langle 6, [\,]\rangle \\ 2 & \mapsto & \langle 7, [\,]\rangle \\ 3 & \mapsto & \langle 7, [\,]\rangle \end{bmatrix} \right\rangle \\ 1 & \mapsto & \langle 7, [\,]\rangle \\ 2 & \mapsto & \langle 6, [\,]\rangle \end{bmatrix} \right\rangle$$

S-term displayed as a well-founded 2D tree

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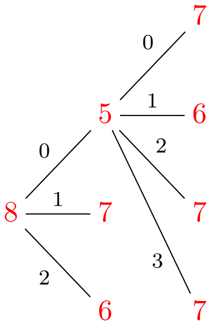

- Vertical dimension for tupling.
- Horizontal dimension for internal structure.
- Root at the left.

Building S-terms and ordinals

To build an S-term, we take a symbol $i \in I$ and a K_i-tuple of S-terms.

This is definite.

To build an ordinal, we take any (transitive) set of ordinals, however big.

Not definite.

Reduced Broad Infinity

First step: the constructors

We want Begin $\in \mathcal{T}$ and Make : $\mathcal{T}^2 \to \mathcal{T}$ so that Make is injective and never yields Begin.

$$\begin{array}{rcl} \text{Begin} & \stackrel{\text{def}}{=} & \emptyset \\ \text{Make}(x,y) & \stackrel{\text{def}}{=} & \{\{x\},\{x,y\}\} \end{array}$$

A reduced broad signature F is a function sending each x

to a set Fx, its arity.

Reduced Broad Infinity (continued)

Let F be a reduced broad signature.

A set X is F-inductive when .

- Begin $\in X$
- for any $x \in X$ and Fx-tuple $(a_k)_{k \in Fx}$ within X, we have $\mathsf{Make}(x, (a_k)_{k \in Fx}) \in X$.

A set of all F-broad numbers is a minimal (and therefore least) F-inductive set.

Axiom scheme of Reduced Broad Infinity: For any reduced broad signature F, there's a set of all F-broad numbers.

F-broad numbers form a class

Easy There's a class of all F-broad numbers, i.e. a least F-inductive class.

F-broad numbers form a class

Easy There's a class of all F-broad numbers, i.e. a least F-inductive class.

This is an example of an introspectively generated class C, where we test $x \in C$ by looking at the $\in$-descendants of x.

(There are other examples, e.g. Ord.)

So Reduced Broad Infinity can be stated thus:
The class rBroad(F) is a set.

Example of an F-broad number

The arity of $\mathsf{Make}(\mathsf{Begin}, [\,])$ is $\{0, 1\}$.
Everything else has arity $\emptyset$.

$$\mathsf{Make}(\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, [\,]), \begin{bmatrix} 0 & \mapsto & \mathsf{Begin} \\ 1 & \mapsto & \mathsf{Make}(\mathsf{Begin}, [\,]) \end{bmatrix}), [\,])$$

F-broad number displayed as a well-founded 3D tree

$$\mathsf{Make}(\mathsf{Make}(\mathsf{Make}(\mathsf{Begin},[\,]),\begin{bmatrix}0 & \mapsto & \mathsf{Begin} \\ 1 & \mapsto & \mathsf{Make}(\mathsf{Begin},[\,])\end{bmatrix}),[\,])$$

- Vertical dimension for tupling.
- Horizontal dimension for Make.
- Depth dimension for internal structure.
- Root at the front.
- Begin-marked leaves at the rear.

Building S-terms and ordinals

To build an S-term, we take a symbol $i \in I$ and a K_i-tuple of S-terms.

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To build an ordinal, we take any (transitive) set of ordinals, however big.

Not definite.

To build an F-broad number, we take either Begin or an F-broad number x and Fx-tuple $[a_k]_{k \in Fx}$ of F-broad numbers.

Definite? I think so!

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To build an F-broad number, we take either Begin or an F-broad number x and Fx-tuple $[a_k]_{k \in Fx}$ of F-broad numbers.

Definite? I think so!

But justification comes from the left and the rear.

First step: the constructors

We want Start $\in \mathcal{T}$ and Build : $\mathcal{T}^3 \to \mathcal{T}$ so that Build is injective and never yields Start.

$$\begin{aligned} \mathsf{Start} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Build}(x,y,z) &\stackrel{\text{def}}{=} \{\{x\},\{x,\{\{y\},\{y,z\}\}\}\} \end{aligned}$$

A broad signature G is a function sending each x to a signature Gx.

Broad Infinity (continued)

Let G be a broad signature.

A set X is G-inductive when

- Start $\in X$
- for any $x \in X$ with $Gx = (K_i)_{i \in I}$, and any $i \in I$ and K_i-tuple $[a_k]_{k \in K_i}$ within X, we have Build$(x, i, [a_k]_{k \in K_i}) \in X$.

A set of all G-broad numbers is a minimal (and therefore least) G-inductive set.

Axiom scheme of Broad Infinity: For any broad signature G, there's a set of all G-broad numbers.

Broad Infinity vs Reduced Broad Infinity

Broad Infinity implies Reduced Broad Infinity

and conversely, using Excluded Middle.

For intuitionistic set theory (IZF), it might be that Broad Infinity is stronger.

Broad Infinity is more complicated, but (in my view) equally intuitive.

It would be strange to accept Reduced Broad Infinity but not Broad Infinity.

Literary interlude

Literary interlude

I look at the changing sea and sky
And try to picture Infinity

(Noel Coward)

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I look at the changing sea and sky
And try to picture Infinity

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2021 update

I look at all the trees outside
And picture Broad Infinity

Generation of sets and families

Given a class C, we consider

- Subset of C generated by a rubric
- Family within C generated by a rubric
- Subset of C generated by a broad rubric
- Family within C generated by a broad rubric.

Example: rubric $\mathcal{R}$ on $\mathbb{N}$

Intuition the rubric tells you when to accept an element of $\mathbb{N}$.

- Rule 0. Arity $= \{0, 1\}$, sends $[m_0, m_1] \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1. Arity $= \emptyset$, sends $[\,] \mapsto (2p)_{p \geqslant 50}$.

Elements accepted by the rubric

- 100 has derivation $\langle 1, [], 50 \rangle$.
- 102 has derivation $\langle 1, [], 51 \rangle$.
- 402 has derivations $\langle 0, [\langle 1, [], 50 \rangle, \langle 1, [], 50 \rangle], 202 \rangle$ and $\langle 0, [\langle 1, [], 50 \rangle, \langle 1, [], 51 \rangle], 200 \rangle$.
- 7 has no derivation, so it is not accepted.

Rubric on a class C

A rule $\langle K, R\rangle$ on C consists of

- a set K—the arity
- a function R sending each K-tuple $[a_k]_{k\in K}$ within C to a family $(y_p)_{p\in P}$.

A rubric on C is a family of rules $(\langle K_i, R_i\rangle)_{i\in I}$, indexed by a set.

Rubric generating a set or family

Let $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ be a rubric on a class C.

Set generated by $\mathcal{R}$

A minimal ($\therefore$ least) subset X of C that is $\mathcal{R}$-inductive:

- for $i \in I$ and tuple $[a_k]_{k \in K_i}$ within X, with $R[a_k]_{k \in K_i} = (y_p)_{p \in P}$, and $p \in P$, we have $y_p \in X$.

Intuition X consists of all the elements obtained from $\mathcal{R}$.

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Intuition X consists of all the elements obtained from $\mathcal{R}$.

Family generated by $\mathcal{R}$

A minimal ($\therefore$ least) family $(x_m)_{m \in M}$ that is $\mathcal{R}$-inductive:

- for $i \in I$ and $g : K_i \to M$, with $R_i(x_{gk})_{k \in K_i} = (y_p)_{p \in P}$, and any $p \in P$, we have $\langle i, g, p \rangle \in M$ and $x_{\langle i,g,p \rangle} = y_p$.

Intuition M is the set of derivations, and $m \in M$ is derivation of x_m.

Example: broad rubric on $\mathbb{N}$

Intuition Once an element is accepted, it triggers a rubric.

The basic rubric is as follows.

- Rule 0. Arity $= \{0,1\}$, sends $[m_0, m_1] \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1. Arity $= \emptyset$, sends $[] \mapsto (2p)_{p \geqslant 50}$.

The rubric triggered by 7 is as follows.

- Rule 0. Arity $= \{0,1\}$, sends $[m_0, m_1] \mapsto (m_0 + m_1 + 500p)_{p \geqslant 9}$.

The rubric triggered by 100 is as follows.

- Rule 0. Arity $= \{0,1,2\}$, sends $[m_0, m_1, m_2]$
  $\mapsto (m_0 + m_1 m_2 + p)_{p \geqslant 17}$.
- Rule 1. Arity $= \emptyset$, sends $[] \mapsto (p)_{p \geqslant 1000}$.
- Rule 2. Arity $= \{0\}$, sends $[m_0] \mapsto (m_0 + p)_{p \geqslant 4}$.

The rubric triggered by any other natural number is empty.

Elements accepted by $\mathcal{B}$

Define Basic : $\mathcal{T}^3 \to \mathcal{T}$ and Trigger : $\mathcal{T}^4 \to \mathcal{T}$, injective and disjoint.

- Basic$(1, [], 50)$ is a derivation of 100.
- Basic$(1, [], 51)$ is a derivation of 102.
- Trigger(Basic$(1, [], 50), 2,$ [Basic$(1, [], 51)], 5)$ is a derivation of 107.
- 7 has no derivation, so it is not accepted.

Broad rubric $\mathcal{B}$ on a class C

The broad rubric $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ consists of

- the basic rubric $\mathcal{B}_0$
- for each $x \in C$, a triggered rubric $\mathcal{B}_1(x)$.

Broad rubric generating a set or family

Let $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ be a broad rubric on a class C.

Set generated by $\mathcal{B}$

A minimal subset X of C that is $\mathcal{B}$-inductive

- is $\mathcal{B}_0$-inductive
- for $x \in X$, is $\mathcal{B}_1(x)$-inductive.

Family generated by $\mathcal{B}$

A minimal family $(x_m)_{m \in M}$ that is $\mathcal{B}$-inductive

- [. . .] then $\mathsf{Basic}(i, g, p) \in M$ and $x_{\mathsf{Basic}(i,g,p)} = y_p$.
- [. . .] then $\mathsf{Trigger}(m, i, g, p) \in M$ and $x_{\mathsf{Trigger}(m,i,g,p)} = y_p$.

Generation principles

Set Generation principle

Every rubric on a class generates a set.

Family Generation principle

Every rubric on a class generates a family.

Broad Set Generation principle

Every broad rubric on a class generates a set.

Broad Family generation principle

Every broad rubric on a class generates a family.

Preliminary questions

- Does every rubric or broad rubric generate a class?

  Tricky question.

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- Does every rubric or broad rubric generate a class?

  Tricky question.

- Does every broad rubric generate a large family $(x_m)_{m \in M}$?

  Yes, these are examples of introspectively generated large families.

  So Broad Family Generation can be stated thus:
  For a broad rubric $\mathcal{S}$, the large family that it generates is a family.

Theorems

- Family Generation.
- Assuming AC Set Generation.
- Broad Infinity $\Leftrightarrow$ Broad Family Generation.
- Assuming AC Broad Infinity $\Leftrightarrow$ Broad Set Generation.

Family Generation $\Rightarrow$ Set Generation, using AC

Given a rubric $\mathcal{R} = (\langle R_i, K_i \rangle)_{i \in I}$ on C.

If $(x_m)_{m \in M}$ is the family generated by $\mathcal{R}$,

then the set of derivable elements, i.e. $\{x_m \mid m \in M\}$,

is the set generated by $\mathcal{R}$.

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Proof

Given $i \in I$ and a K_i-tuple of derivable elements,
choose a derivation for each.

Use these to build a bigger derivation.

Weaker than AC suffices

For a set X, a cover is a family $(L_x)_{x \in X}$ of inhabited sets.

The WISC axiom says that for every set X there is a weakly initial set of covers. [Moerdijk, Palmgren, Rathjen, van den Berg]

A global WISC function sends every set X to a weakly injective set of covers.

This suffices to prove Broad Family Generation $\Rightarrow$ Broad Set Generation.

Application: Grothendieck universes

A Grothendieck universe extending a set X is a transitive set $\mathfrak{U}$ such that

- $X \subseteq \mathfrak{U}$.
- $\mathbb{N} \in \mathfrak{U}$.
- For every set of sets $\mathcal{A} \in \mathfrak{U}$, we have $\bigcup \mathcal{A} \in \mathfrak{U}$.
- For every set $A \in \mathfrak{U}$, we have $\mathcal{P}A \in \mathfrak{U}$.
- For every set $K \in \mathfrak{U}$ and K-tuple $[a_k]_{k \in K}$ within $\mathfrak{U}$, we have $\{a_k \mid k \in K\} \in \mathfrak{U}$.

Broad Set Generation implies that every set generates a Grothendieck universe. (The "axiom of universes".)

Application: Tarski-style universes

A Tarski-style universe extending a family of sets $(B_a)_{a \in A}$ is a family of sets $(D_m)_{m \in M}$ such that

- For all $a \in A$, we have $\mathrm{embed}(a) \in M$ with $D_{\mathrm{embed}(a)} = B_a$.
- We have $\mathrm{zero} \in M$ with $D_{\mathrm{zero}} = \emptyset$.
- ...
- For any $m \in M$ and function $g : D_m \to M$, we have $\mathrm{pi}(m, g) \in M$ with $D_{\mathrm{pi}(m,g)} = \prod_{k \in D_m} D_{g(m)}$.

Broad Family Generation implies that every family of sets generates a Tarski-style universe.

Ordinals

Every set A has a Lindenbaum number $\aleph^*(A)$,

the set of order-types of well-ordered partial partitions of A.

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Properties

1. A does not have a strictly increasing $\aleph^*(A)$-chain of subsets.
2. Any A-indexed family of ordinals has range with order-type $< \aleph^*(A)$.

Generation by transfinite induction

Let $\mathcal{R}$ be a rubric or broad rubric on a class C.

We define an increasing chain $(X_\gamma)_{\gamma \in \mathsf{Ord}}$ of subsets of C.

At 0, take the empty set.
At limit ordinals, take the union.
At $\alpha + 1$, take those elements obtained from applying a rule to elements in X_α.

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Stabilization

- If the chain stabilizes at γ i.e. $X_\gamma = X_{\gamma+1}$, then X_γ is the set generated by $\mathcal{R}$.
- Conversely, if $\mathcal{R}$ generates a set A, then the chain stabilizes before $\aleph^*(A)$, by Lindenbaum property (1).

Ordinal generation principles

Note that ordinal = transitive subset of Ord.

Blass Generation, follows from Set Generation

For any α, there is least regular ordinal $\kappa \geqslant \alpha$.

Equivalent to Blass's Axiom.

Equivalent to Set Generation, by property (2).

Jorgensen Generation, instance of Broad Set Generation

For any $F : \mathsf{Ord} \to \mathsf{Ord}$,
there is least regular $\kappa \geqslant F$, i.e. $\gamma < \kappa$ implies $F\gamma \leqslant \kappa$.

Equivalent to Ord-is-Mahlo.

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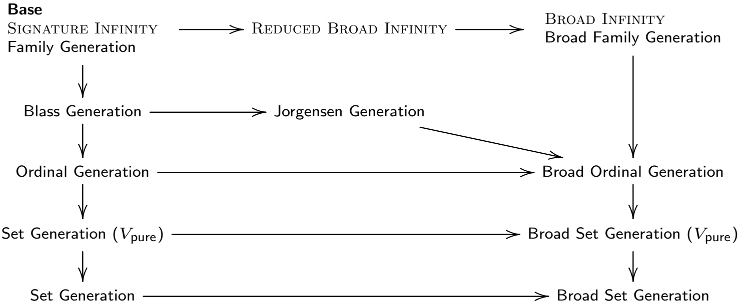
Diagram of subsystems, for the CZF school

The Base Theory doesn't assume Truth Value Set, and allows urelements and non-well-founded-sets.



V_{pure} is the class of pure well-founded sets.

Diagram of subystems, for the IZF school

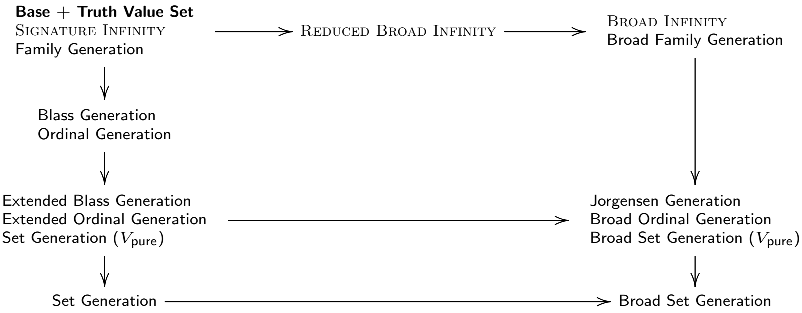


Diagram of subsystems, for choiceless mathematics

Base + **Boolean Truth** = **ZFAN**
SIGNATURE INFINITY
Family Generation

→

REDUCED BROAD INFINITY
BROAD INFINITY
Broad Family Generation

↓ (from Base + Boolean Truth = ZFAN)

Blass Generation
Ordinal Generation
Set Generation
BLASS'S AXIOM

→

↓ (from Reduced Broad Infinity)

Jorgensen Generation
Broad Ordinal Generation
Broad Set Generation
ORD-IS-MAHLO

Diagram of subsystems, for mathematics with AC

Base + AC = ZFCAN		REDUCED BROAD INFINITY
SIGNATURE INFINITY		BROAD INFINITY
Family Generation		Broad Family Generation
Blass Generation	$\longrightarrow$	Jorgensen Generation
Ordinal Generation		Broad Ordinal Generation
Set Generation		Broad Set Generation
BLASS'S AXIOM		ORD-IS-MAHLO

Conclusion

- Broad Infinity is a new axiom scheme. Hopefully you find it intuitive.
- It is equivalent to Broad Family Generation.
- It is equivalent to many other schemes if Excluded Middle, global WISC function or AC is assumed.
- Everything works in the presence of urelements and/or non-well-founded sets.
- The Infinity story and the Broad Infinity story are somewhat analogous.

Further Work

- Broad version of Gitik's result?
- Connection to Induction-Recursion in type theory.
- Is Broad Infinity validated in models? (Cf. Rathjen)
- Is a global WISC function available in models?
- Problems with unrestricted quantification