Real closures of ω_1-like models of PA

David Marker

Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
marker@uic.edu

On-line meeting celebrating Jim Schmerl's 85th birthday.

I first met Jim in 1979 in the long running Connecticut Logic Seminar that rotated between Yale, Wesleyan and UConn.

My favorite theorem of Jim's

Question:. Are there recursively saturated models of PA that are the Skolem hulls of a sequence of indiscernibles?

Theorem (Abramson and Knight)

Yes!

Theorem (Schmerl)

Every *countable recursively saturated model of PA is the Skolem hull of a set of indiscernibles!!*

- Jim's proof uses combinatorial methods of Nešetřil and Rödl.

Real closures of models of PA

Suppose $\mathcal{M} \models \mathrm{PA}$.
Let $Q(\mathcal{M})$ be the fraction field of $\mathcal{M}$ and let $R(\mathcal{M})$ be the real closure of $Q(\mathcal{M})$.

Theorem (D'Aquino–Knight–Starchenko)

If K is a non-archimedean real closed field and $R \subset K$ is an integer part of K with $R \models \mathrm{PA}$, then K is recursively saturated.
In particular, if $\mathcal{M} \models \mathrm{PA}$, then $R(\mathcal{M})$ is recursively saturated.

Corollary

If $\mathcal{M}, \mathcal{N}$ are countable nonstandard models of PA, then $R(\mathcal{M}) \cong R(\mathcal{N})$ if and only if $SS(\mathcal{M}) = SS(\mathcal{N})$.

i.e., passing from $\mathcal{M}$ to $R(\mathcal{M})$ all information is lost except $SS(\mathcal{M})$.

$SS(\mathcal{M})$"$=$" largest subfield of $\mathbb{R}$ embedding into $R(\mathcal{M})$.

ω_1-like models

Corollary (DKS)

If $\mathcal{M}, \mathcal{N} \models \mathrm{PA}$ are countable and $SS(\mathcal{M}) = SS(\mathcal{N})$, then $R(\mathcal{M}) \cong R(\mathcal{N})$.

Question: If $\mathcal{M}, \mathcal{N}$ are ω_1-like models of PA with $SS(\mathcal{M}) = SS(\mathcal{N})$, is $R(\mathcal{M}) \cong R(\mathcal{N})$?

Theorem (Marker–Steinhorn)

If $\mathcal{M}, \mathcal{N} \models \mathrm{PA}$ are ω_1-like and $SS(\mathcal{M}) = SS(\mathcal{N})$, then the value groups of $R(\mathcal{M})$ and $R(\mathcal{N})$ are isomorphic.

Theorem (Marker–Schmerl–Steinhorn)

There are $(\mathcal{M}_\alpha : \alpha < 2^{\aleph_1})$ recursively saturated ω_1-like models of PA such that $\mathcal{M}_\alpha \equiv_{\infty,\omega_1} \mathcal{M}_\beta$ but $R(\mathcal{M}_\alpha) \not\cong R(\mathcal{M}_\beta)$ for all $\alpha \neq \beta$.

Main Theorem–Ingredient 1: Conservative Extensions

Definition

Let $\mathcal{M} \models \mathrm{PA}$. We say that an elementary end extension $\mathcal{M} \prec_e \mathcal{N}$ is *conservative* if $X \cap \mathcal{M}$ is definable in $\mathcal{M}$ whenever $X \subseteq \mathcal{N}$ is definable in $\mathcal{N}$.

Two classical facts on models of PA.

Lemma

i) Every model of PA *has a conservative elementary end extension.*
ii) Every countable model of PA *has a non-conservative elementary end extension.*

Ingredient 2: Rather Classless Models

Definition

If $\mathcal{M} \models \mathrm{PA}$ we say that $X \subseteq \mathcal{M}$ is a *class* if $\{x \in X : x < a\}$ is definable for all $a \in \mathcal{M}$.
We say that $\mathcal{M} \models \mathrm{PA}$ is *rather classless* if every class is definable.

Lemma

Suppose that

$$\mathcal{M}_0 \prec_e \mathcal{M}_1 \prec_e \ldots \mathcal{M}_\alpha \prec_e \ldots, \text{ for } \alpha < \omega_1$$

is a continuous chain of countable elementary end extensions. If

$$\{\alpha < \omega_1 : \mathcal{M}_{\alpha+1} \text{ is a conservative extension of } \mathcal{M}_\alpha\} \text{ is stationary,}$$

then $\bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ *is rather classless.*

Ingredient 3: Last chance to realize types

Lemma

Suppose that $\mathcal{M} \prec_e \mathcal{M}' \prec_e \mathcal{M}''$ are models of PA. Then any nonprincipal 1-type q over $R(\mathcal{M})$ realized in $R(\mathcal{M}'')$ is already realized in $Q(\mathcal{M}')$.

By o-minimality q is determined by a cut in the ordering of $R(\mathcal{M})$.

Suppose $a \in R(\mathcal{M}'')$ realizes q. Without loss of generality $0 < a < 1$.

If $n > \mathcal{M}$, $b \in R(\mathcal{M}'')$ and $|a - b| < \frac{1}{n}$, then b also realizes q.

Pick $d \in \mathcal{M}'$ such that $d > \mathcal{M}$.
Find $c \in \mathcal{M}''$ such that

$$\frac{c}{d} < a < \frac{c+1}{d}.$$

Since $c < d$ and $\mathcal{M}' \prec_e \mathcal{M}''$, $c \in \mathcal{M}'$ and $\frac{c}{d}$ realizes q.

Ingredient 4: Scott completions

Definition

Let F be an ordered field. An initial segment $I \subseteq F$ is *Dedekindean* if $I + \epsilon \not\subseteq I$ for all $\epsilon > 0$.

For example, consider $F = \mathbb{Q}(\epsilon)$ where ϵ is infinitesimal.

- $I = \{x \in F : x < 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^n$ for some $n\}$ is Dedekindian
- $I = \{x \in F : x < \pi\}$ is not Dedekindean. (ϵ infinitesimal)
- $I = \{x \in F : x < n$ for some $n \in \mathbb{N}\}$ is not Dedekindian. ($\epsilon = 1$)

Definition

An ordered field F is *Scott complete* if every Dedekindian initial segment has a supremum in F.

Theorem (Scott '69)

If F is an ordered field there is a unique Scott complete ordered field $\widehat{F} \supseteq F$ such that F is dense in $\widehat{F}$.

Scott completions of models of PA

If $\mathcal{M} \models \mathrm{PA}$ let $SC(\mathcal{M})$ be the Scott completion of $Q(\mathcal{M})$.

Lemma (Schmerl '85)

If $\mathcal{M} \models \mathrm{PA}$, then $SC(\mathcal{M})$ is real closed.

Let $A \subseteq \mathcal{M}$ be a class and let

$$I_A = \{x \in Q(\mathcal{M}) : x \leq \sum_{a \in A, a \leq b} \frac{1}{2^{a+1}} \text{ for some } b \in \mathcal{M}\}.$$

- I_A is Dedekindian.
- $A \mapsto I_A$ is a bijection between classes of $\mathcal{M}$ and Dedekidian initial segments in $Q(\mathcal{M}) \cap [0, 1]$.

Basic Construction

For $\mathcal{M}_0 \models \mathrm{PA}$ countable and $X \subseteq \omega_1$ stationary construct a continuous chain of countable models

$$\mathcal{M}_0 = \mathcal{M}_0(X) \prec_e \mathcal{M}_1(X) \prec_e \ldots \prec_e \mathcal{M}_\alpha(X) \prec_e \ldots \text{ for } \alpha < \omega_1$$

such that $\mathcal{M}_{\alpha+1}$ is a conservative extension of $\mathcal{M}_\alpha$ if and only if $\alpha \in X$. Let $\mathcal{M}(X) = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha(X)$.

Since $\mathcal{M}(X)$ is rather classless all Dedekindian cuts are definable, so $|SC(\mathcal{M}(X))| = \aleph_1$.

Choose a filtration $S_0(X) \subseteq S_1(X) \subseteq \cdots \subseteq S_\alpha(X) \subseteq \ldots \ \alpha < \omega_1$ of $SC(\mathcal{M}(X))$ where each $S_\alpha(X)$ is countable.

Main Lemma

Lemma

Suppose X, Y are stationary and $Y \setminus X$ is stationary. Then $R(\mathcal{M}(X))$ is not isomorphic to $R(\mathcal{M}(Y))$.

Any isomorphism $\sigma : R(\mathcal{M}(X)) \to R(\mathcal{M}(Y))$ would extend to an isomorphism of the Scott completions $\sigma : SC(\mathcal{M}(X)) \to SC(\mathcal{M}(Y))$.

We can find $\alpha \in Y \setminus X$ such that:
i) σ restricts to an isomorphism between $R(\mathcal{M}_\alpha(X))$ and $R(\mathcal{M}_\alpha(Y))$;
ii) $S_\alpha(X)$ is real closed and σ is an isomorphism onto $S_\alpha(Y)$;
iii) $S_\alpha(X)$ is the set of all Dedekindian initial segments of $Q(\mathcal{M}(Y))$ definable over $\mathcal{M}_\alpha(X)$ and the same for $S_\alpha(Y)$;
iv) $S_\alpha(X) \cap R(\mathcal{M}(X)) = R(\mathcal{M}_\alpha(X))$ and the same for $S_\alpha(Y)$.

- There is $A \subset \mathcal{M}_{\alpha+1}(X)$ definable such that $A \cap \mathcal{M}_\alpha(X)$ is not definable. Let $b \in \mathcal{M}_{\alpha+1}(X) \setminus \mathcal{M}_\alpha(X)$ and

$$x = \sum_{a \in A, a < b} \frac{1}{2^{a+1}}.$$

- $q = \mathrm{tp}(x/\mathcal{M}_\alpha(X))$ corresponds to an undefinable Dedekindian cut so not in $S_\alpha(\mathcal{M})$.
- $\sigma(q)$ is realized in $R(\mathcal{M}(Y))$ and hence in $Q(\mathcal{M}_{\alpha+1}(Y))$.
- Since $\mathcal{M}_{\alpha+1}(Y)$ is a conservative extension of $\mathcal{M}_\alpha(Y)$, $\sigma(q)$ is in $S_\alpha(Y)$.

  This contradicts the fact that σ is an isomorphism between $S_\alpha(X)$ and $S_\alpha(Y)$ sending $R(\mathcal{M}_\alpha(X))$ to $R(\mathcal{M}_\alpha(Y))$.

Final Conclusions

- Usual tricks allow us to find a family $(X_\alpha : \alpha < 2^{\aleph_1}\}$ of stationary subsets of ω_1 with $X_\alpha \setminus X_\beta$ stationary for $\alpha \neq \beta$.
- Adding a predicate for a partial satisfaction class Γ and working in PA^* allows us to assume all $\mathcal{M}_\alpha$ are recursively saturated.
- (Kossak) If $\mathcal{M} \equiv \mathcal{N}$ are ω_1-like recursively saturated models of PA and $SS(\mathcal{M}) = SS(\mathcal{N})$, then $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$.

Happy Birthday Jim!