LINKING DESCRIPTIVE SET THEORY TO SYMBOLIC DYNAMICS

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The original problem

In Barcelona in 1993 Moira Chas told me of an iteration question in compact metric spaces which appeared to involve countable ordinals.

[**ACS**] LL. ALSEDÀ, M. CHAS and J. SMÍTAL. On the structure of the ω-limit sets for continuous maps of the interval. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), no. 9, 1719–1729. MR 2000**i**:37047

Let $\mathcal{X}$ be a Polish space and let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous function. For $k \in \omega$ we write f^k for the k^{th} iterate of f, so that for each $x \in \mathcal{X}$, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. Then define the ω-limit set $\omega_f(x)$ to be the set

$$\left\{ y \in \mathcal{X} \;\middle|\; \exists \text{ (strictly) increasing } \alpha : \omega \to \omega \text{ with } \lim_{n\to\infty} f^{\alpha(n)}(x) = y \right\}.$$

REMARK $\omega_f(x)$ is a closed subset of $\mathcal{X}$.

Define an operator Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

Then starting from a given point $a \in \mathcal{X}$, define a transfinite sequence:

$$\begin{aligned} A^0(a,f) &= \omega_f(a) \\ A^{\beta+1}(a,f) &= \Gamma_f(A^\beta(a,f)) \\ A^\lambda(a,f) &= \bigcap_{\nu<\lambda} A^\nu(a,f) \qquad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

By elementary analysis, $A^0(a,f) \supseteq A^1(a,f) \supseteq A^2(a,f)\ldots$; and indeed for all ordinals $\alpha < \beta$, $A^\alpha(a,f) \supseteq A^\beta(a,f)$.

Thus if we make the following

DEFINITION $\theta(a,f) =_{\mathrm{df}}$ the least ordinal θ with $A^\theta(a,f) = A^{\theta+1}(a,f)$,

we know that $\theta(a,f)$ is well defined; and for all $\delta \geqslant \theta$, $A^\delta(a,f) = A^\theta(a,f)$. edskip

DEFINITION We write $A(a,f)$ for this final set $A^{\theta(a,f)}(a,f)$. We call $A(a,f)$ the *abode*, and the ordinal $\theta(a,f)$ the *score* of the point a under f.

The question raised in 1993 was to investigate the possible behaviour of the function $\theta(a,f)$: what are its possible values ?

DEFINITION The *escape set* or *boundary* is the union over all ordinals β of the set of those points in $\omega_f(a)$ eliminated at stage β of the iteration:

$$E(a,f) =_{\mathrm{df}} \bigcup_\beta \big(A^\beta(a,f) \smallsetminus A^{\beta+1}(a,f)\big).$$

Here $X \smallsetminus Y$ is the set-theoretic difference $\{x \mid x \in X \text{ and } x \notin Y\}$.

DEFINITION For $x \in E(a,f)$, we write $\beta(x,a,f)$ for the unique β with $x \in A^\beta(a,f) \smallsetminus A^{\beta+1}(a,f)$.

Thus $E(a,f) = \omega_f(a) \smallsetminus A(a,f)$. We say that points in $A(a,f)$ *abide*, and points in $E(a,f)$ *escape*.

My set-theoretical interest was aroused, and led initially to two papers.

[**2a**] A. R. D. Mathias. Delays, recurrence and ordinals. *Proc. London Math. Soc.* (3) **82** (2001) 257–298.

[**2b**] A. R. D. Mathias, Recurrent points and hyperarithmetic sets, in *Set Theory, Techniques and Applications*, Curaçao 1995 and Barcelona 1996 conferences, edited by C. A. Di Prisco, Jean A. Larson, Joan Bagaria and A. R. D. Mathias, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998, 157–174.

My initial progress was made by switching attention from the sets $\omega_f(x)$ to a binary relation which I called *attack*.

DEFINITION x f-*attacks* y, in symbols $x \curvearrowright_f y$, if for every open n'h'd G of y the set $\{n \mid f^n(x) \in G\}$ is infinite.

PROPOSITION *(i) if $x \curvearrowright_f y$ and $y \curvearrowright_f z$ then $x \curvearrowright_f z$.*
(ii) If $x \curvearrowright_f a$ then $x \curvearrowright_f f(a)$ and $f(x) \curvearrowright_f a$.

REMARK $y \in \omega_f(x) \iff x \curvearrowright_f y$.

REMARK The advantage of changing to work with the relation $\curvearrowright_f$ is that for $\mathcal{X}$ second-countable and f continuous, the binary relation $\curvearrowright_f$ is G_δ; so that the Kunen–Martin theorem of descriptive set theory may be applied to show that $\beta(x, a, f)$ is always countable, and hence that $\theta(a, f)$ never exceeds ω_1.

Background reading

[**3**] Y. N. Moschovakis. *Descriptive set theory.* (North Holland, 1980).

[**4**] A. S. Kechris. *Classical descriptive set theory.* Graduate Texts in Mathematics 156, (Springer, 1995).

[**5**] C. Dellacherie, Un cours sur les ensembles analytiques, in: *Analytic Sets* by C. A. Rogers *et al.*, Academic Press, London etc 1980, pp 183–316.

For next week's talk:

[**K**] K. KUNEN, *Some points in* $\beta\mathbb{N}$, Math. Proc. Cam. Phil. Soc. 80 (1976) 385–398

[**B**] Andreas BLASS, *Ultrafilters: where topological dynamics = algebra = combinatorics.* Topology Proc. 18 (1993), 33–56.

More recent work

[**2c**] A. R. D. Mathias, Analytic sets under attack, *Math. Proc. Cambridge Phil. Soc.* **138** (2005) 465–485.

[**2d**] A. R. D. Mathias, Choosing an attacker by a local derivation, *Acta Universitatis Carolinae - Math. et Phys.*, **45**(2004) 59–65.

[**2e**] A. R. D. Mathias, A scenario for transferring high scores, *Acta Universitatis Carolinae - Math. et Phys.*, **45** (2004) 67–73.

Yet more recent work

[**6a**] C. Delhommé. Transfer of scores to the shift's attacks of Cantor space.

[**6b**] C. Delhommé. Representation in the shift's attacks of Baire space. [*formerly* On embedding transitive relations in that of shift-attack.]

[**6c**] C. Delhommé. Completeness properties of the relation of attack.

Related to next week's talk:

[**M**] T.K. Subrahmonian MOOTHATHU, *Syndetically proximal pairs*, J. Math. Anal. Appl. **379** (2011) 656–663

My five publications prove theorems in a Polish (therefore metrisable) but not necessarily compact space. Blass in his classic paper *Ultrafilters: Where topological dynamics = algebra = combinatorics* uses ultrafilters to good effect when the space is compact but not necessarily metrisable. Next week I shall try to extend this use of ultrafilters to incompact spaces in exploring the notion of *uniform attack*.

General notation

These two talks apply set-theoretic ideas to a problem of analysis, and therefore our notation will draw on that of two mathematical traditions. Thus we usually denote the set $\{0, 1, 2, \ldots\}$ of *natural numbers* by ω, though occasionally by $\mathbb{N}$; this visual distinction allows us to write ω^n for the ordinal power and $\mathbb{N}^n$ for the set of n-tuples of natural numbers.

$\mathbb{N}^+$ is the set $\{1, 2, 3, \ldots\}$ of *positive integers*: in Definition 4·3 the difference between $\mathbb{N}$ and $\mathbb{N}^+$ is important.

On a space, such as Baire space, comprising all sequences of length ω of members of some set, we define the shift function $\mathfrak{s}$ thus:

$$\mathfrak{s}(\zeta)(n) = \zeta(n+1) \text{ for } n \geqslant 0.$$

Here we return to normal set-theoretic convention by considering the domain of such sequences to be $\omega = \{0, 1, 2, \ldots\}$.

We write $\circledcirc$ for the empty sequence: technically of course it is the same as the empty set, which we write as $\varnothing$; and also the same as the number zero, which we write as 0, since set-theorists customarily identify each natural number n with the set $\{0, 1, \ldots n-1\}$.

We denote by ${}^{<\omega}X$ the set of finite sequences of points in the set X, including the empty sequence.

When s is a finite sequence, we write $\ell h(s)$ for its length, so that $s = \langle s(0), s(1), \ldots s(\ell h(s)-1)\rangle$. We also write $\ell(s)$ for its last element, $s(\ell h(s)-1)$. Concatenation is denoted by $^\frown$, so $\ell h(s^\frown\langle p\rangle) = \ell h(s)+1$.

Set-theoretic preliminaries

The next few slides are taken from §1 of *Delays*.

Our constructions will be based on well-founded relations of a particular kind, *well-founded trees of finite sequences.*

For a non-empty set X — for example, let $X = \omega$ — we define a relation on the set ${}^{<\omega}X$.

1·0 DEFINITION $t \preccurlyeq s \iff_{\text{df}} t$ is an extension of s; $t \prec s \iff_{\text{df}} t$ is an proper extension of s; $s \succcurlyeq t \iff_{\text{df}} s$ is an initial segment of t; $s \succ t \iff_{\text{df}} s$ is a proper initial segment of t.

1·1 REMARK Thus $s \succcurlyeq t \iff t \preccurlyeq s$, and so on. $\circledcirc$ has no proper initial segments, but is itself a proper initial segment of every finite sequence of positive length. Note that longer sequences are lower in this ordering.

1·2 DEFINITION A *tree* in this talk will mean a subset of ${}^{<\omega}X$ which is *closed under shortening* in the sense that $s \succcurlyeq t \in T \implies s \in T$. Thus if T is non-empty it will contain the empty sequence $\circledcirc$. We shall refer to the members of T as its *nodes*.

1·3 DEFINITION A tree T is *well-founded* if whenever C is a non-empty subset of T there is an $s \in C$ such that no $t \prec s$ is in C. Such an s is termed a T-*minimal* element of C.

1·4 REMARK If X has a well-ordering, as is the case with the two main examples, or if we assume DC, then saying that T is well-founded is equivalent to the requirement that there be no infinite path through T: that is, that there is no function $f : \omega \to X$ such that for each n, the finite sequence $f \restriction n =_{\mathrm{df}} \langle f(0), f(1), \dots, f(n-1) \rangle$ is in T.

Given a well-founded tree T that is closed under shortening we may define a *rank function* ϱ_T on it by recursion:

$$\varrho_T(s) = \sup\{\varrho_T(t) + 1 \mid t \in T \,\&\, t \prec s\}$$

Some comments on this definition: if T consists solely of the empty sequence, $\varrho_T(\circledcirc) = 0$. For any non-empty well-founded T there will be by definition of well-foundedness nodes of T with no proper extension in T; such nodes, which we term *bottom nodes* of the tree, will have rank 0. Should ϱ_T not be defined for all nodes of the tree, we may by well-foundedness find a node s such that ϱ_T is not defined at s but is defined for each proper extension of s. But then the recipe tells us how to proceed to define ϱ_T at s.

The above illustrates the process of *definition by induction* on a well-founded tree. There is also available a method of *proof by induction* on a well-founded tree:

1·5 PROPOSITION *Let T be a well-founded tree, and $\Phi(s)$ some property. If $\forall s_{\in T}\big[(\forall t \prec s\ \Phi(t)) \Longrightarrow \Phi(s)\big]$ then $\forall s_{\in T}\ \Phi(s)$.*

That may be proved by supposing $\{s \mid \neg\Phi(s)\}$ to be non-empty, considering a T-minimal element thereof, and reaching a contradiction. It may also be proved by using the rank function ϱ_T and considering a counterexample s with $\varrho_T(s)$ minimal. Just such an argument proves the following

1·6 PROPOSITION *Let T be a well-founded tree and $s \in T$. For each $\nu < \varrho_T(s)$ there is a $t \prec s$ with $\varrho_T(t) = \nu$.*

Linking escape to well-foundedness

The slides of this section are taken from §2 of *Delays*.

We introduce the trees we shall use to calculate $\beta(b)$ for $b \in E(a, f)$. We shall define for our fixed a and for each $b \in \mathcal{X}$ a tree T^a_b of finite sequences and show using DC that $b \in A(a, f) \iff T^a_b$ is ill-founded.

2·0 DEFINITION For $b \in \mathcal{X}$, set

$$T^a_b \;=_{\mathrm{df}}\; \big\{s \in {}^{<\omega}\mathcal{X} \mid \ell h(s) > 0 \Longrightarrow \big(s(0) = b \;\&\; \forall i\!:<\ell h(s)\ (a \curvearrowright s(i)) \;\&\; \forall i\!:<\ell h(s){-}1\ (s(i+1) \curvearrowright s(i))\big)\big\}.$$

Note that if $t \succ s \in T^a_b$, then $t \in T^a_b$, so that T^a_b is closed under shortening. Our definition is of most interest when $b \in \omega_f(a)$, since

$$b \notin \omega_f(a) \iff T^a_b = \{\circledcirc\}.$$

2·1 LEMMA (DC) *$b \in A(a,f) \iff \exists$ an infinite sequence $\langle x_i | i < \omega\rangle$ such that $\forall i_{\in\omega}\ a \curvearrowright x_i$ and*

$$b = x_0 \curvearrowleft x_1 \curvearrowleft x_2 \curvearrowleft \ldots \quad .$$

Proof : given such a sequence, one checks easily by induction on ξ that each of its members is in $A^\xi(a,f)$, hence is in $A(a,f)$; in particular $b = x_0$ is in $A(a,f)$. If no such sequence exists for a given b, then by DC the tree T^a_b will be well-founded under $\prec$, and hence we may define a rank function $\varrho = \varrho^a_b$ mapping T^a_b to the ordinals by

$$\varrho^a_b(s) = \sup\{\varrho^a_b(s^\frown\langle r\rangle) + 1 \mid r \in \mathcal{X} \ \&\ s^\frown\langle r\rangle \in T^a_b\}.$$

and show by induction on ξ that $\varrho^a_b(s) = \xi \implies \ell(s) \notin A^{\xi+1}(a,f)$: hence $b \notin A^{\varrho^a_b(\langle b\rangle)+1}(a,f)$.

2·2 Corollary (DC) *For* $b \in \omega_f(a)$, $b \in E(a, f) \iff T^a_b$ *is well-founded.*

2·4 Proposition *For each* $b \in E(a, f)$, $\varrho^a_b(\langle b \rangle) < \omega_1$.

2·5 Corollary $\theta \leqslant \omega_1$

Proof : Each b in $E(a, f)$ leaves the A-sequence at the countable stage $\varrho^a_b(\langle b \rangle) + 1$. Hence by stage ω_1 all those points that are to escape have already done so. ⊣ (2·5)

Points at the end of a path

We had earlier this easy

PROPOSITION *(i) if* $x \curvearrowright_f y$ *and* $y \curvearrowright_f z$ *then* $x \curvearrowright_f z$.
(ii) If $x \curvearrowright_f a$ *then* $x \curvearrowright_f f(a)$ *and* $f(x) \curvearrowright_f a$.

which now yields this invaluable

PROPOSITION *Let* f *be a continuous map of a Polish space* $\mathcal{X}$ *into itself, and suppose that we have an infinite sequence of points* b_i, *with* $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots \curvearrowleft_f b$. *Then we can choose integers* n_i, *(increasing if we wish), such that putting* $y_i = f^{n_i}(b_i)$, *the* y_i *form a Cauchy sequence converging to a point* y *with* $b \curvearrowright_f y \curvearrowright_f y \curvearrowright_f b_i$ *for each* i.

Proof : in these circumstances $f^n(b_j) \curvearrowright b_i$ for $j > i$ and arbitrary n. $\dashv$

DEFINITION Let $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots$ be an infinite path descending in the relation $\curvearrowright_f$. We say that a point y *lies at the end of the path* if it satisfies two conditions:

(i) there are numbers n_i such that $y = \lim_{i\to\infty} f^{n_i}(b_i)$;

(ii) for each i, $y \curvearrowright_f b_i$.

PROPOSITION *If both y and z are at the end of the same path, then $y \curvearrowright_f z \curvearrowright_f y$; in particular all points at the end of a given path are recurrent and attack each other.*

Proof : True because z attacks each b_i, hence attacks each $f^{n_i}(b_i)$; hence attacks y; and the situation is symmetric. $\dashv$

3: maximal recurrent points

3·0 DEFINITION A *recurrent* point is a b such that $b \curvearrowright b$.

It has long been known that the existence of recurrent points is neither certain nor impossible:

3·1 EXAMPLE Let $\mathcal{X} = \mathbb{R}$, and $f(x) \equiv x + 1$. Then f has no recurrent points.

3·2 THEOREM (AC) *Let $\mathcal{X}$ be a compact Polish space and $f : \mathcal{X} \longrightarrow \mathcal{X}$ continuous. Then recurrent points exist: indeed each $x \in \mathcal{X}$ attacks at least one recurrent point.*

3·3 REMARK The above use of AC could be reduced to an application of DC by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem, which appears as Theorem 8F.10 on page 526 of [14].

We may use the following lemma since in a metric space second countability and separability are equivalent conditions.

3·4 Lemma (*AC*) *In a second countable space $\mathcal{X}$ there can exist neither a strictly descending sequence $\langle C_\nu \mid \nu < \omega_1 \rangle$ nor a strictly ascending sequence $\langle D_\nu \mid \nu < \omega_1 \rangle$ of non-empty closed subsets of $\mathcal{X}$.*

Proof : given a descending counter-example in a space with countable basis $\{N_s \mid s \in \omega\}$, pick $p_\nu \in C_\nu \setminus C_{\nu+1}$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap C_{\nu+1}$ empty. There will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\delta \in C_\delta \cap N_{s_\delta} \subseteq C_{\nu+1} \cap N_{s_\nu} = \varnothing$, a contradiction. In the ascending case, pick $p_\nu \in D_{\nu+1} \setminus D_\nu$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap D_\nu$ empty. Again there will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\nu \in D_{\nu+1} \cap N_{s_\nu} \subseteq D_\delta \cap N_{s_\delta} = \varnothing$, another contradiction. $\dashv$ (3·4)

3·5 Remark Hausdorff in §27 of his book *Mengenlehre* [6] proves with a beautiful argument that, more generally, there cannot be an uncountable sequence, whether strictly increasing or strictly decreasing, of sets that are simultaneously F_σ and G_δ. That may be used to prove that in Baire space, neither the set $\{\beta \mid \beta \curvearrowright_{\mathfrak{s}} \beta\}$ nor for any ε the set $\{\beta \mid \beta \curvearrowright_{\mathfrak{s}} \varepsilon\}$ is $\mathbf{\Sigma^0_2}$, and therefore the relation $R_{\mathfrak{s}}$ cannot be, either; see Proposition 7·6 of [13] for the details, but note that in two places in the proof, β_n is printed instead of $\beta \restriction n$.

Proof of 3·2: We know that each $\omega_f(x)$ is a closed set, which, by sequential compactness is non-empty, and that if $y \in \omega_f(x)$ then $\omega_f(y) \subseteq \omega_f(x)$. Start from x, and set $C_0 = \omega_f(x)$. We shall define a shrinking sequence of closed sets all of the form $\omega_f(z)$.

If $C_\xi = \omega_f(x_\xi)$ ask if there is a $y \in C_\xi$ such that $\omega_f(y)$ is a proper subset of C_ξ: if not, then x_ξ is recurrent (in a strong sense, indeed). If there is, pick some such and call it $x_{\xi+1}$, and take $C_{\xi+1} = \omega_f(x_{\xi+1})$.

At limit stages, take the intersection, call it C'_λ: by compactness it will be non-empty. Pick x_λ in it. Then for each $\nu < \lambda$ $x_\nu \curvearrowright x_\lambda$; so $\omega_f(x_\lambda) \subseteq C'_\lambda$. Set $C_\lambda = \omega_f(x_\lambda)$ and continue.

By the Lemma this process stops before stage ω_1: we have then reached a z such that $\forall w_{\in \omega_f(z)}\, w \curvearrowright z$: since $\omega_f(z)$ is non-empty, such a z is evidently recurrent, and is attacked by our original x. ⊣ (3·2)

3·6 REMARK In such a case z, or the set $\omega_f(z)$,is called *minimal*.

Characterising the abode by recurrent points

The following result shows that provided not every point in $\omega_f(a)$ escapes, recurrent points exist. We emphasize that the space is not assumed to be compact. The apparent use of the Axiom of Choice is avoidable.

3·7 THEOREM *Let $\mathcal{X}$ be a complete separable metric space, $f : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous map, and a, x arbitrary points in $\mathcal{X}$. Then*

$$x \in A(a, f) \iff \exists b \; a \curvearrowright b \curvearrowright b \curvearrowright x.$$

Maximal recurrent points.

3·18 PROPOSITION *Given $\mathcal{X}$, f, and a, suppose that for all i $a \curvearrowright z_{i+1} \curvearrowright z_i \curvearrowright \ldots \curvearrowright z_0$. Then there are natural numbers $m_0 < m_1 < \ldots$ such that setting $y_i = f^{m_i}(z_i)$, the sequence (y_i) is convergent with limit b, say, and $b \curvearrowright y_i$ for each i. It follows that b is recurrent, and that for all i, $a \curvearrowright b \curvearrowright z_i$ and $\omega_f(z_i) = \omega_f(y_i)$.*

3·19 REMARK Note that if the points z'_i form a second set satisfying the hypothesis of the Proposition, with $\forall i\ z_i \curvearrowright z'_i \curvearrowright z_i$, and y'_i, b' are the outcome of repeating the argument, then

$$\forall i\ b \curvearrowright z_{i+1} \curvearrowright z'_{i+1} \curvearrowright y'_i \ \&\ b' \curvearrowright z'_{i+1} \curvearrowright z_{i+1} \curvearrowright y_i$$

and so $b \curvearrowright b' \curvearrowright b$.

3·20 DEFINITION Call a point b *maximal recurrent in* $\omega_f(a)$ if $a \curvearrowright b \curvearrowright b$ and whenever $a \curvearrowright c \curvearrowright c \curvearrowright b$, then $b \curvearrowright c$.

With the help of the axiom of choice the above proposition yields the following

3·21 COROLLARY (*AC*) *If d is a recurrent point in $\omega_f(a)$, then there is a point b which is maximal recurrent in $\omega_f(a)$ with $a \curvearrowright b \curvearrowright d$.*

Proof : set $d_0 = d$. If d_0 is not maximal in $\omega_f(a)$, pick d_1 with $a \curvearrowright d_1 \curvearrowright d_1 \curvearrowright d_0 \not\curvearrowright d_1$; if d_1 is not maximal, continue. Proposition 3·18 tells us that our construction can be continued at countable limit ordinals. If we never encounter a maximal recurrent point, then our construction will yield for every countable ordinal ν a recurrent point d_ν with $a \curvearrowright d_\zeta \curvearrowright d_\nu \not\curvearrowright d_\zeta$ for $\nu < \zeta < \omega_1$. But then the sequence $\langle \omega_f(d_\nu) \mid \nu < \omega_1 \rangle$ will form a strictly increasing sequence of closed sets of order type ω_1, contradicting Lemma 3·4. ⊣ (3·21)

3·22 Remark Again, that use of AC could be reduced to an application of DC by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem.

3·23 Remark We could also formulate the notion of a *maximal recurrent* point in the space $\mathcal{X}$ as a whole, without reference to a particular point a; the same argument will prove that if recurrent points exist, so do maximal ones. In a case such as the shift function acting on Baire space, the maximal recurrent points will be simply be those whose orbit is dense in the whole space.

3·24 Remark Note that it follows from Theorem 3·15 that each point in the abode A is in the closure of the set of recurrent points. The converse need not hold, as we shall see later; and thus the abode is not necessarily identical with the set of *non-wandering points* studied by earlier writers such as Birkhoff, which exactly equals that closure.

Building points of large countable score

These slides are taken from §4 of *Delays*;
for more general embeddings, see Delhommé [**6b**].

4: Long delays in Baire space

Explicit construction of well-founded trees

For any ordinal η we can uniformly build a tree of height that ordinal:

4·0 DEFINITION For an arbitrary ordinal η let T_η be the set of all strictly descending sequences of ordinals less than η.

T_η is, naturally, a well-founded tree. We include the empty sequence $\circledcirc$ in each T_η as its topmost point.

4·1 PROPOSITION *For each* η, $\varrho_{T_\eta}(\circledcirc) = \eta$.

Evidently when η is countable T_η will be isomorphic to a tree $U \subseteq {}^{<\omega}\omega$; it will be convenient instead to find a tree, isomorphic to T_η, that is a subset of the set $\mathcal{S}$ we now define.

4·2 Definition Let $\mathcal{S}$ be the set of finite strictly increasing sequences of odd prime numbers (excluding 1). We count $\circledcirc$, the empty sequence, as a member of $\mathcal{S}$.

4·3 With an eye to applications in §7, we show, more generally, that given a countable *linear* ordering $(X, <)$, we may uniformly define a map ψ, from the set of decreasing finite sequences of members of X to the set of increasing finite sequences of odd primes, which preserves the end-extension relation. Note that $(X, <)$ need not be a well-ordering; it might for example be the set of rationals under the Euclidean order.

Let $h : X \xrightarrow{1-1} \omega$. Using h we can assign to each $x \in X$ an bijection (usually not order-preserving !) g_x of $\{y \in X \mid y < x\}$ and either some finite n or ω.

We set $\psi(\circledcirc) = \circledcirc$. We map sequences of length 1 to sequences of odd primes of length 1, using h composed with an enumeration p_i of odd primes: $\psi(\langle x \rangle) = \langle p_{h(x)} \rangle$.

Now suppose we have already defined $\psi(s)$, where $s \neq \circledcirc$. Let x be the least element of s, and let p_j be the largest element of $\psi(s)$. If $t = s^\frown \langle y \rangle$ where $y < x$, set $\psi(t) = \psi(s)^\frown \langle p_{j+1+g_x(y)} \rangle$.

The plan of attack

We explore and exploit the possibility of embedding countable well-founded trees into the relation $\curvearrowright$.

4·4 Lemma *Let* $f : \mathcal{X} \longrightarrow \mathcal{X}$ *be continuous, and let* T *be a non-empty well-founded tree with top-most point* $\circledcirc$*. Suppose that we have points* x_T *and* x_s *(for* $s \in T$*) in the space* $\mathcal{X}$ *such that for all* s *and* t *in* T*,* $x_T \curvearrowright x_s$ *and if* $s \prec t$ *then* $x_s \curvearrowright x_t$*. Then for each* $s \in T$*,* $x_s \in A^{\varrho_T(s)}(x_T, f)$*.*

Proof : following 1·6, let $r \prec s \Longrightarrow x_r \in A^{\varrho_T(r)}(x_T, f)$. Then $x_s \in \omega_f(x_T) \cap \bigcap\{A^{\varrho_T(r)+1}(x_T, f) \mid r \prec s\}$ which by 1·7 and 0·2 equals $A^{\varrho_T(s)}(x_T, f)$. ⊣ (4·4)

We would like to have $\forall s_{\in T}\, x_s \notin A^{\varrho_T(s)+1}$, so that $\alpha < \beta \leqslant \varrho_T(\circledcirc) + 1 \implies A^\alpha \supsetneq A^\beta$ and we should then have $\theta(x_T, f) > \varrho_T(\circledcirc)$. The next lemma gives further conditions on our points x_T, x_s which will make that happen. We present this argument in an abstract setting in terms of a *nearness relation* between points, which, to emphasize its possibly asymmetric character, we write as $b \triangleright_f x$, or, more conveniently, as $b \triangleright x$, which may be read as "*b is near to x*".

4·5 Example In our first application we shall take $b \triangleright x$ to mean that for some $n \geqslant 0$, $f^n(x) = b$; plainly that is liable to be asymmetric. In our second application we shall take $b \triangleright x$ to have the plainly symmetrical meaning that for some non-negative n, m, $f^m(b) = f^n(x)$. In both we shall have $b \triangleright b$ for every b.

4·6 Lemma *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\circledcirc$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ and a relation $b \triangleright x$ between points such that whenever $s \in T$ & $x_T \curvearrowright c \curvearrowright b \triangleright x_s$, then for some $r \in T$ with $r \prec s$, $c \triangleright x_r$. Then for $b \in \omega_f(x_T)$ and $s \in T$,*

$$x_T \curvearrowright b \triangleright x_s \Longrightarrow b \notin A^{\varrho(s)+1}(x_T, f).$$

Proof : write $\Phi(s)$ for "$x_T \curvearrowright b \triangleright x_s \Longrightarrow b \notin A^{\varrho(s)+1}(x_T, f)$". We suppose inductively that $\forall r \prec s\ \Phi(r)$ and prove $\Phi(s)$. So suppose $x_T \curvearrowright c \curvearrowright b \triangleright x_s$. By assumption, $\exists r_{\in T}\ [r \prec s\ \&\ c \triangleright x_r]$; by $\Phi(r)$, $c \notin A^{\varrho_T(r)+1}$, so $c \notin A^{\varrho_T(s)}$. As c was arbitrary, $b \notin A^{\varrho_T(s)+1}$, and we have proved that $\Phi(s)$ holds. $\dashv$ (4·6)

These lemmata lead to the following general result:

4·7 Theorem *Let $\mathcal{X}$ be a complete separable metric space, let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\circledcirc$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ and a relation $b \triangleright_f x$ between points of $\omega_f(x_T)$ such that for all s, t in T, writing $\curvearrowright$ for $\curvearrowright_f$ and $\triangleright$ for $\triangleright_f$,*

(4·7·0) $x_T \curvearrowright x_s$;

(4·7·1) $s \prec t \Longrightarrow x_s \curvearrowright x_t$;

(4·7·2) $x_s \triangleright x_s$;

(4·7·3) $x_T \curvearrowright c \curvearrowright b \triangleright x_s \Longrightarrow c \triangleright x_r$ *for some* $r \in T$ *with* $r \prec s$.

Then $\theta(x_T, f) > \varrho_T(\circledcirc)$.

In the fourth section of the paper *Delays* we construct for particular well-founded trees T of arbitrary countable rank points x_s and x_T in Baire space satisfying the hypotheses of the above theorem, and find that $\theta(x_T, \mathfrak{s}) = \varrho_T(\circledcirc) + 1$: we then modify our examples to obtain in each space points z_T with $\theta(z_T, f) = \varrho_T(\circledcirc)$.

In the fifth section of *Delays*, we construct points in certain compact spaces to which the theorem applies, though the corresponding modification proves troublesome, and the transfer theorems of Delhommé give a welcome simplification of the discussion and improvement of the results.

In the sixth section of *Delays* we essay applications to certain ill-founded trees.

Examples of long delays in the Baire space

4·8 DEFINITION *Baire space*, $\mathcal{N}$, is $\{b \mid b : \omega \longrightarrow \omega\}$; topologically it is the product of $\aleph_0$ copies of ω, each with the discrete topology.

4·9 DEFINITION The *shift function* $\mathfrak{s} : \mathcal{N} \longrightarrow \mathcal{N}$ is defined by $\mathfrak{s}(b)(n) = b(n+1)$ for $b : \omega \longrightarrow \omega$.

REMARK Some call that the *backward* shift: it does lose information.

The construction below together with the remarks at the end of the section will prove the following:

4·10 THEOREM *Let $\mathcal{N}$ be Baire space ω^ω, and $\mathfrak{s}$ the shift operation. Then for each countable ordinal ζ there is a point $a \in \mathcal{N}$ such that $\theta(a, \mathfrak{s}) = \zeta$.*

Our plan in *Delays* was this: to each $s \in \mathcal{S}$ we defined a point $x_s \in \mathcal{N}$; we wrote $b \triangleright x$ to mean that b is *a finite shift of* x in the sense that $b = \mathfrak{s}^n(x)$ for some $n \geqslant 0$ and $x \in \mathcal{N}$; then for each well-founded $T \subseteq \mathcal{S}$ we defined a point x_T so that the points x_T and x_s for $s \in T$ together with the relations $\curvearrowright = \curvearrowright_{\mathfrak{s}}$ and $\triangleright$ satisfy the hypotheses of Theorem 4·7.

4·11 Definition We write $u \sqsubset x$ to mean that the non-empty finite sequence u occurs as a segment of the infinite sequence x.

4·12 Lemma *If $x \curvearrowright y$ and $u \sqsubset y$ then u occurs infinitely often as a segment of x.*

Some open problems: more next week

PROBLEM (Cummings) *Is score a* Π^1_1 *norm ?*

PROBLEM *What are the possible scores under s of recursive members of* $\mathcal{N}$ *?*

We know that there are recursive $\beta \in \mathcal{N}$ where the score of β is any given recursive ordinal, or the first non-recursive ordinal [**2b**] or the first uncountable ordinal [**2c**]: are there any others ?

A possibility might be ω_1^L.

The notion of a *uniformly recurrent* point has been much studied:

PROBLEM *Is there a reasonable definition of "x uniformly attacks y".*

Preparations for a point of uncountable score

The slides of this section are taken from §3 of *Analytic sets under attack*.

Finite trees and paths

We write $\ell h(u)$ for the length of a finite sequence u.

3·0 DEFINITION $\mathcal{F} =_{\text{df}} \{u \mid u$ a non-empty finite sequence

$$\big(u(1), u(2), \ldots, u(\ell h(u))\big)$$

of natural numbers $u(i)$ with $0 \leqslant u(i) < i$ for $1 \leqslant i \leqslant \ell h(u)\}$.

3·1 REMARK Contrary to habitual practice among set theorists, the terms of u are indexed by $1, \ldots, \ell h(u)$ rather than $0, \ldots, \ell h(u) - 1$.

For $1 \leqslant k \leqslant \ell h(u)$ we write $u_{\leqslant k}$ for the sequence $\big(u(1), \ldots, u(k)\big)$; that will be an element of $\mathcal{F}$.

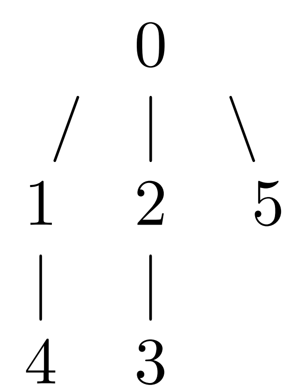
3·3 DEFINITION If $u = \big(u(1), u(2), \ldots, u(\ell h(u))\big) \in \mathcal{F}$, a *positive u-sequence* is a non-empty finite sequence $s = (p_1, \ldots, p_\ell)$ with $1 \leqslant p_1 < p_2 < \cdots < p_\ell \leqslant \ell h(u)$, so that $\ell = \ell h(s)$ and $p_\ell = \max s$; we further require that $u(p_1) = 0$, and for $1 \leqslant i < \ell h(s)$, $u(p_{i+1}) = p_i$.

The *u-sequences* are the positive u-sequences and the empty sequence, which we write as $\circledcirc$.

As above, we write $s_{\leqslant k}$ for the sequence $(p_1, \ldots, p_k)$, where $1 \leqslant k \leqslant \ell h(s)$; that too will be a positive u-sequence. Further, we interpret $s_{\leqslant 0}$ as the empty sequence, $\circledcirc$.

We read an element u of $\mathcal{F}$ as coding a finite downwards-branching tree with 0 as the unique top point and $u(i)$ immediately above i for each i with $1 \leqslant i \leqslant \ell h(u)$.

3·4 Example Let $u \in \mathcal{F}$ be the sequence (0,0,2,1,0). Then, with our convention on indexing, $u(1) = 0; u(2) = 0; u(3) = 2; u(4) = 1; u(5) = 0$, so we read u as coding this tree:



Thus the u-sequences are ⊚, (1), (2), (5), (1,4), and (2,3), 3·5 We shall build our point in a space of infinite sequences of *symbols*, of which there will be three kinds, *recorders, predictors* and *markers.* Certain symbols will contain information that is either an element u of $\mathcal{F}$—such symbols will be called *recorders*, because they contain information about the recent past of the infinite sequence of symbols under consideration—or else a pair of finite sequences s, u where $u \in \mathcal{F}$ and s is a positive u-sequence—such symbols will be called *predictors* because they contain information about the near future of that infinite sequence. Nothing is required of the third kind of symbol, the *markers*, save that there be a countable infinity of them and that they be all distinct from each other and from all recorders and predictors.

It is extremely important that, from the point of view of the shift function that we shall apply, each symbol is a single object; and, to give visual emphasis to that point, we shall use square brackets $[,\ ]$ to encase each individual symbol, whereas we shall use pointed brackets $\langle,\ \rangle$, to encase finite or infinite sequences of symbols.

We shall associate to each recorder and each predictor two natural numbers, its *weight* and its *height*.

3·6 DEFINITION A *recorder* is an object $[u]$ where u is in $\mathcal{F}$. Its *weight* is 0 and its *height* is the length $\ell h(u)$ of u as a member of $\mathcal{F}$.

3·7 DEFINITION A *predictor* is an object $[s;u]$ where $u \in \mathcal{F}$ and s is a positive u-sequence. s will be called the *path* of the predictor $[s;u]$, and u its *tree*. The predictor's *weight* is the length of its path, and its *height* is the length of its tree.

3·8 REMARK The weight of $[s;u]$ is not greater than its height.

3·9 DEFINITION We say that s is *tight* in u, or that u *tightly contains* s, if s is a u-sequence and $\max s = \ell h(u)$. In the contrary case we shall use the words *loose* and *loosely*. We may indeed define the *looseness of* u *over* s as $\ell h(u) - \max s$.

3·10 For each $u \in \mathcal{F}$ and each u-sequence s we shall define a finite sequence z^u_s of symbols. Our definition will proceed by a mode of induction that will also be used in proving our theorem, which we shall call *double induction*. To spell the method out in greater detail: we first consider the case $s = \circledcirc$. Then we suppose that $m \geqslant 1$ and that we have already treated all pairs u, s with s a u-sequence of length $< m$. On that supposition, we take an s of length m, and consider all $u \in \mathcal{F}$ for which s is a u-sequence, starting with those u for which $\ell h(u) = \max s$, and then progressively treating longer u; thus for given s we proceed by induction on the looseness of u over s. The following convention will be useful.

3·11 Definition We write s' for the sequence s with its last element removed—so that if s is of length 1, $s' = \circledcirc$—and we write u' for u with its last element removed.

We proceed to our definition of z_s^u by double induction, and first treat the case of $s = \circledcirc$.

3·12 Definition For $u \in \mathcal{F}$,

$$z_{\circledcirc}^{u} =_{\mathrm{df}} \left\langle [u_{\leqslant 1}], [u_{\leqslant 2}], \ldots, [u_{\leqslant \ell h(u)-1}], [u] \right\rangle.$$

3·13 Remark The length of $z_{\circledcirc}^{u}$ equals that of u.

3·14 Example

$$z_{\circledcirc}^{(0,0,2,1,0)} = \left\langle [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \right\rangle$$

Now for $u \in \mathcal{F}$ and s a positive u-sequence we shall define z^u_s.

3·15 Definition

$$z^u_s \;=_{\rm df} \begin{cases} \langle [s;u] \rangle ^\frown z^u_{s'} & \text{if } \max s = \ell h(u); \\ z^{u'}_s {}^\frown \langle [s;u] \rangle ^\frown z^u_{s'} & \text{if } \max s < \ell h(u). \end{cases}$$

The first clause handles the case that u tightly contains s, and the second the cases when $\ell h(u)$ is strictly greater than $\max s$.

3·16 Remark Note that $[s;u]$ occurs only once in z^u_s; we shall refer to it as the *peak* of z^u_s. It is the only symbol in z^u_s with sum of weight and height equal to $\ell h(s) + \ell h(u)$.

We give several examples to illustrate that definition.

3·17 Example If s is of length 1, then $z^u_s = \langle [s;u] \rangle ^\frown z^u_\circledcirc$ if $\max s = \ell h(u)$ and $z^u_s = z^{u'}_s {}^\frown \langle [s;u] \rangle ^\frown z^u_\circledcirc$ otherwise.

3·18 Example If u is the sequence (0,0,2,1,0), then $z^u_{(5)}$ is

$$\big\langle [(5);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle,$$

a sequence of six symbols, whereas $z^u_{(2)}$ is

$$\begin{aligned}\big\langle [(2);(0,0)], [(0)], [(0,0)], [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\ [(2);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\ [(2);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle,\end{aligned}$$

which has eighteen, of which the heights, in order, are 2, 1, 2; 3, 1, 2, 3; 4, 1, 2, 3, 4; 5, 1, 2, 3, 4, 5.

$$z_{(1)}^{(0)} = \Big\langle [(1);(0)], [(0)] \Big\rangle;$$

$$z_{(1)}^{(0,0)} = \Big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)] \Big\rangle;$$

$$z_{(1)}^{(0,0,2)} = \Big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)] \Big\rangle;$$

$$z_{(1)}^{(0,0,2,1)} = \Big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)] \Big\rangle;$$

$$z_{(1)}^{(0,0,2,1,0)} = \Big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(1);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \Big\rangle.$$

$$
\begin{aligned}
z^{(0,0,2,1)}_{(1,4)} &= \Big\langle[(1,4);(0,0,2,1)]\Big\rangle^\frown z^{(0,0,2,1)}_{(1)}\\
&= \Big\langle[(1,4);(0,0,2,1)],\\
&\qquad [(1);(0)],[(0)],[(1);(0,0)],[(0)],[(0,0)],\\
&\qquad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)],\\
&\qquad [(1);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)]\Big\rangle;
\end{aligned}
$$

$$
\begin{aligned}
z_{(1,4)}^{(0,0,2,1,0)} &= z_{(1,4)}^{(0,0,2,1)}{}^\frown\Big\langle[(1,4);(0,0,2,1,0)]\Big\rangle^\frown z_{(1)}^{(0,0,2,1,0)} \\
&= \Big\langle[(1,4);(0,0,2,1)], \\
&\qquad\quad [(1);(0)],[(0)], \\
&\qquad\quad [(1);(0,0)],[(0)],[(0,0)], \\
&\qquad\quad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
&\qquad\quad [(1);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)], \\
&\qquad [(1,4);(0,0,2,1,0)], \\
&\qquad\quad [(1);(0)],[(0)], \\
&\qquad\quad [(1);(0,0)],[(0)],[(0,0)], \\
&\qquad\quad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
&\qquad [(1);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)], \\
&\quad [(1);(0,0,2,1,0)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],[(0,0,2,1,0)]\Big\rangle.
\end{aligned}
$$

$$
\begin{aligned}
z_{(2,3)}^{(0,0,2)} &= \langle [(2,3);(0,0,2)] \rangle^\frown z_{(2)}^{(0,0,2)}; \\
z_{(2,3)}^{(0,0,2,1)} &= z_{(2,3)}^{(0,0,2)\frown} \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)}; \\
z_{(2,3)}^{(0,0,2,1,0)} &= z_{(2,3)}^{(0,0,2,1)\frown} \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)} \\
&= z_{(2,3)}^{(0,0,2)\frown} \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)\frown} \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)}
\end{aligned}
$$

which equals

$$\begin{aligned}
&\Big\langle [(2,3);(0,0,2)],\\
&\qquad\quad [(2);(0,0)],[(0)],[(0,0)],\\
&\qquad\quad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)],\\
&\qquad [(2,3);(0,0,2,1)],\\
&\qquad\quad [(2);(0,0)],[(0)],[(0,0)],\\
&\qquad\quad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)],\\
&\qquad\quad [(2);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],\\
&\qquad [(2,3);(0,0,2,1,0)],\\
&\qquad\quad [(2);(0,0)],[(0)],[(0,0)],\\
&\qquad\quad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)],\\
&\qquad\quad [(2);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],\\
&\qquad\quad [(2);(0,0,2,1,0)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],[(0,0,2,1,0)]\Big\rangle.
\end{aligned}$$

3·19 EXAMPLE Suppose that $3 + \max t = \ell h(v)$. Let $v_i = v_{\leqslant i+\max t}$, so that $v_0 = v_{\leqslant \max t}$ and $v_3 = v$.

Then z_t^v is

$$\langle [t; v_0] \rangle ^\frown z_{t'}^{v_0} {}^\frown \langle [t; v_1] \rangle ^\frown z_{t'}^{v_1} {}^\frown \langle [t; v_2] \rangle ^\frown z_{t'}^{v_2} {}^\frown \langle [t; v] \rangle ^\frown z_{t'}^{v},$$

which has precisely the four predictors shown of weight equal to the length of t; all other predictors in z_t^v will be of lesser weight.

Here is a first example of proof by double induction:

3·20 PROPOSITION *If s is not $\circledcirc$, then the first symbol of z_s^u is the predictor $[s; u_{\leqslant \max s}]$.*

Proof : If u tightly contains s, $z_s^u = \langle [s; u] \rangle ^\frown z_{s'}^u$ of which the first symbol is $[s; u]$, which equals $[s; u_{\leqslant \max s}]$. Otherwise $z_s^u = z_s^{u'} {}^\frown \langle [s; u] \rangle ^\frown z_{s'}^u$, of which the first symbol is that of $z_s^{u'}$, which, by the induction hypothesis, is the predictor $[s; u'_{\leqslant \max s}]$; but that in the context equals $[s; u_{\leqslant \max s}]$.

⊣ (3·20)

Notation for finite sequences

3·21 DEFINITION $t \preccurlyeq s \iff_{\mathrm{df}} t$ is an extension of s; $t \prec s \iff_{\mathrm{df}} t$ is an proper extension of s; $s \succcurlyeq t \iff_{\mathrm{df}} s$ is an initial segment of t; $s \succ t \iff_{\mathrm{df}} s$ is a proper initial segment of t.

3·22 REMARK Thus $s \succcurlyeq t \iff t \preccurlyeq s$, and so on. $\circledcirc$ has no proper initial segments, but is itself a proper initial segment of every finite sequence of positive length. Note that longer sequences are lower in this ordering.

3·23 DEFINITION We shall say that two finite sequences s and t *cohere* if either $s \succcurlyeq t$ or $t \succcurlyeq s$.

Properties of finite sequences

3·24 Proposition *Let u and v be members of $\mathcal{F}$, and let t be both an u-sequence and a v-sequence.*

(i) $\ell h(u) = \ell h(z^u_{\circledcirc})$;

(ii) *for* $\ell \leqslant \ell h(v)$, $z^v_{\circledcirc} \restriction \ell = z^{v\restriction\ell}_{\circledcirc}$;

(iii) $v \prec u \Longrightarrow z^v_t \prec z^u_t$;

(iv) $z^v_t = z^u_t \Longrightarrow v = u$;

(v) $z^v_t \prec z^u_t \Longrightarrow v \prec u$.

Proof of 3·24 (iii): If $t = \circledcirc$, use (ii): otherwise use an earlier instance to note that $z_t^v \prec z_t^{v'} \preccurlyeq z_t^u$.
Proof of 3·24 (iv): Compare peaks.
Proof of 3·24 (v): The peak of z_t^v cannot be in z_t^u, for otherwise $u = v$; whence $z_t^u \succcurlyeq z_t^{v'}$, giving, inductively, $v' \preccurlyeq u$.

3·25 Definition An *m-predictor* is a predictor of weight exactly m. An *m-stretch* is a finite sequence of symbols all of weight at most m.

3·26 Lemma *Let $u \in \mathcal{F}$, s a u-sequence of weight $> m$. Let $x \sqsubseteq z^u_s$ be an m-stretch.*

(i) $x \sqsubseteq z^u_{s'}$;

(ii) *in fact* $x \sqsubseteq z^u_{s_{\leqslant m}}$.

Proof of 3·26 (i)*:* Its weight forbids the peak of z^u_s to lie in x.

Case 1: s is tight in u. Then $z^u_s = \langle [s;u] \rangle ^\frown z^u_{s'}$, whence $x \sqsubseteq z^u_{s'}$.

Case 2: otherwise. Then $z^u_s = z^{u'}_s {}^\frown \langle [s;u] \rangle ^\frown z^u_{s'}$, so either $x \sqsubseteq z^{u'}_s$ or $x \sqsubseteq z^u_{s'}$; if the second alternative is false, we may iterate the first, progressively shortening u till it does tightly contain s, and then apply Case 1. ⊣ (3·26·i)

Proof of 3·26 (ii)*:* By iterating Lemma 3·26 (i), progressively shortening s. ⊣ (3·26·ii)

Indeed we can sharpen that result:

3·27 PROPOSITION *Let x be an m-stretch with all symbols of height at most h. Suppose that $x \sqsubseteq z^u_s$. Then $x \sqsubseteq z^{u_{\leqslant h}}_{s_{\leqslant m}}$.*

Proof : For fixed x by double induction on s and u. If the peak of z^u_s occurs in x, then both the height and weight of x equal those of z^u_s, and then the proposition is trivially true. Otherwise $x \sqsubseteq z^{u'}_s$ or $x \sqsubseteq z^u_{s'}$; in the first case the height is less and in the second the weight. In either case we have a reduction to an earlier instance of the induction. ⊣ (3·27)

3·28 LEMMA *The recorders in z^u_s are those in $z^u_{\circledcirc}$: namely non-empty initial segments of u. Hence any two recorders in z^u_s cohere.*

Proof : By applying Proposition 3·27 to 0-stretches of length 1. ⊣ (3·28)

3·29 LEMMA *If* $s \succcurlyeq t$ *and* t *is a* u*-sequence, then* z^u_s *is a final segment of* z^u_t*; if* $s \succ t$*, that final segment is immediately preceded by the predictor* $[s^+; u]$*, where* $s^+ = t_{\leqslant \ell h(s)+1}$.

Proof : Write $t_0 = t$, and progressively write $t_{k+1} = t'_k$ till we reach $t_n = s$. If $n = 0$ the Lemma is trivial; if $n > 0$, then we remark that for each k, $z^u_{t_k}$ ends in $z^u_{t_{k+1}}$ which is preceded by $[t_k; u]$; finally note that $t_{n-1} = t_{\leqslant \ell h(s)+1}$. ⊣ (3·29)

3·30 LEMMA *if* $u \succcurlyeq v$ *and* s *is a* u*-sequence, then* $z^u_s \succcurlyeq z^v_s$*; if* $u \succ v$*, the term in* z^v_s *after that occurrence of* z^u_s *is* $[s; u^+]$*. where* $u^+ = v_{\leqslant \ell h(u)+1}$.

Proof : The first part is Proposition 3·24 (iii) rephrased; the second part holds if $v' = u$, and stays true for longer v by an easy induction, as then $u \succ v' \succ v$. ⊣ (3·30)

3·31 LEMMA *If* $[s;u]$ *occurs in* z_t^v *then* $s \succcurlyeq t$ *and* $u \succcurlyeq v$.

Proof : By a double induction on t and v. The lemma is true if $[s;u] = [t;v]$. Otherwise $[s;u]$ occurs in $z_{t'}^v$ or, provided t is loose in v, in $z_t^{v'}$; in either case we have a reduction to an earlier instance of the induction, to which we then link either the fact that $t' \succ t$ or that $v' \succ v$. ⊣ (3·31)

3·32 LEMMA *An occurrence of* $[s;u]$ *in* z_t^v *is followed by the whole of* $z_{s'}^u$.

Proof : By a similarly structured induction on t and v. ⊣ (3·32)

3·33 LEMMA *In any* z_s^u *the immediate successor of an* m*-predictor is a symbol of weight* $m - 1$.

Proof : Immediate from the definition if $m = 1$; by Proposition 3·20 otherwise. ⊣ (3·33)

3·34 Lemma *If s is of length $m+1$, $\langle[s;u]\rangle^\frown x$ is a final segment of z^w_s and x is an m-stretch, then $u = w$ and $x = z^u_{s'}$.*

Proof : $[s;w]$ is the last symbol of weight $m+1$ in z^w_s. ⊣ (3·34)

3·35 Proposition *If s is of length $m+1$, x is an m-stretch, and $y =_{\mathrm{df}} \langle[s;u]\rangle^\frown x^\frown\langle[s;v]\rangle \sqsubseteq z^w_r$, then $u = v'$ and $x = z^u_{s'}$.*

Proof by double induction: By Proposition 3·27, we can suppose $r = s$. If $v \neq w$, we have $z^w_s = z^{w'}_s{}^\frown\langle[s;w]\rangle^\frown z^w_{s'}$ and therefore $y \sqsubseteq z^{w'}_s$; thus we may reduce the length of w until $w = v$.

So our proposition is now reduced to the case that $y \sqsubseteq z_s^v$. We then have

$$\langle [s;u] \rangle^\frown x^\frown \langle [s;v] \rangle \sqsubseteq z_s^{v'\frown} \langle [s;v] \rangle^\frown z_{s'}^v;$$

since $[s;v]$ occurs in neither $z_s^{v'}$ nor in $z_{s'}^v$, we may be sure that the last symbol of y occurs as the peak of z_s^v; but then $\langle [s;u] \rangle^\frown x$ forms a final segment of $z_s^{v'}$, so we may apply Lemma 3·34 to infer that $u = v'$ and $x = z_{s'}^u$. ⊣ (3·35)

3·36 Corollary *If* $y = \langle [s; u_1] \rangle^\frown x_1{}^\frown \langle [s; u_2] \rangle^\frown x_2{}^\frown \langle [s; u_3] \rangle \sqsubseteq z_r^w$, *where* s *is of length* $m + 1$ *and both* x_1 *and* x_2 *are* m*-stretches, then* $x_1 \succ x_2$, *and* $\ell h(u_2) = \ell h(u_1) + 1$.

Proof : In the circumstances, $x_1 = z_{s'}^{u_1}$, $x_2 = z_{s'}^{u_2}$, and $u_1 = (u_2)'$.

⊣ (3·36)

3·37 Lemma *If* s *is of length* $m{+}1$, x *is an* m*-stretch, and* $x^\frown \langle [s; v] \rangle \sqsubseteq z_t^w$, *then* x *is a final segment of* $z_s^{v'}$.

Proof : The hypotheses imply, by Proposition 3·27, that $x^\frown \langle [s; v] \rangle \sqsubseteq z_s^v$, in which the only occurrence of $[s; v]$ is the peak; but then x must be a final segment of the preceding sequence, which is $z_s^{v'}$. ⊣ (3·37)

3·38 LEMMA *If the recorder* $[e]$, *of height at least 2, occurs in* z^u_s, *its predecessor is* $[e_{\leqslant \ell h(e)-1}]$; *if of height 1, its predecessor, if any, will be a predictor of weight 1.*

3·39 PROPOSITION *If* $z^u_s(i)$ *and* $z^u_s(i{+}1)$ *are both recorders then* $\ell h(z^u_s(i+1)) = 1 + \ell h(z^u_s(i))$.

3·40 REMARK The unique longest m-stretch in z^u_s is at the end, namely $z^u_{s_{\leqslant m}}$: for if s is of weight m, z^u_s is itself an m-stretch; and if s is of greater weight, the m-stretches in z^u_s are those of $z^u_{s'}$ and, provided s is loose in u, of $z^{u'}_s$. By induction, the unique longest of those are $z^u_{s_{\leqslant m}}$ and $z^{u'}_{s_{\leqslant m}}$, of which two the first is in any case strictly longer. ⊣ (3·40)

3·41 PROPOSITION *Suppose that* $x =_{\text{df}} \big\langle [s;u] \big\rangle^\frown z^u_{s'} \sqsubseteq z^w_r$ *but is not a final segment thereof. Then the first symbol after the segment* x *of* z^w_r *is of the form* $[t;v]$ *where* $v' = u$ *and* $t \preccurlyeq s$*, and if* $t \prec s$ *there will be a later occurrence in* z^w_r *of a symbol of weight that of* s*.*

3·42 REMARK $\big\langle [s;u] \big\rangle^\frown z^u_{s'}$ is a final segment of z^u_s, properly so if and only if s is loose in u.

Towards the proof of Proposition 3·41, we first prove a Lemma to cover the case $s = r$.

3·43 LEMMA $x =_{\text{df}} \big\langle [s;u] \big\rangle^\frown z^u_{s'}$ *is a final segment of* z^w_s *if and only if* $u = w$*.*

Proof : One way is covered by Remark 3·42. For the other, since $z^w_s = z^{w'}_s{}^\frown \big\langle [s;w] \big\rangle^\frown z^w_{s'}$, the peak of z^w_s is its last symbol of weight $\ell h(s)$ and therefore if x is a final segment of z^w_s, the first symbol of x must be that peak, whence $z^u_{s'} = z^w_{s'}$, whence $u = w$. ⊣ (3·43)

Proof of Proposition 3·41*:* We consider s and u to be fixed and do a double induction on r and w.

As always, we have

$$z_r^w = z_r^{w'} {}^\frown \langle [r; w] \rangle {}^\frown z_{r'}^w$$

The hypotheses imply that $r \preccurlyeq s$ and, by Lemma 3·43, that $w \prec u$; hence the peak of z_r^w cannot lie in x, and therefore either $x \sqsubseteq z_{r'}^w$ or $x \sqsubseteq z_r^{w'}$.

If $x \sqsubseteq z_{r'}^w$, then x will not be a final segment of $z_{r'}^w$, and so the induction will apply.

If $x \sqsubseteq z_r^{w'}$, either $w' \prec u$, whence by Lemma 3·43 x is not final in $z_r^{w'}$, and the induction will again apply; or $w' = u$, x is final—again by Lemma 3·43—in $z_r^{w'}$ and the next symbol is $[r; w]$, which is of the desired form $[t; v]$ with $v' = u$ and $t \preccurlyeq s$.

The final clause follows from Lemma 3·33. ⊣ (3·41)

3·44 Proposition *In any z_s^u, if the same symbol, of weight m, occurs twice, then between the two occurrences there must be an occurrence of a symbol of weight $m+1$.*

Proof by double induction: The indicated symbol, that which repeats, cannot be the peak of z_s^u, which occurs only once there. If s is tight in u, the two occurrences must both be in $z_{s'}^u$, and we have reduced to an earlier case.

Otherwise $z_s^u = z_s^{u'} \cap \langle [s;u] \rangle \cap z_{s'}^u$, and there are three possibilities: both occurrences are before the peak, when both lie in $z_s^{u'}$; both lie after, and therefore both lie in $z_{s'}^u$—both times we have a reduction to an earlier case—or one lies before the peak and the other after; but then the proposition is proved, for the peak is of weight greater than m, and, if of weight $> m+1$, will by Lemma 3·33 immediately be followed by symbols of weights declining by 1 at each step, thus reaching a symbol of weight $m+1$ before the second occurrence of the indicated symbol. ⊣ (3·44)

The remaining slides are taken from §4 of *Analytic sets under attack*.

Introducing infinite sequences

We have introduced two of our three kinds of symbol. For the third, the *markers*, we take infinitely many objects $[\mathsf{m}_0], [\mathsf{m}_1], \ldots$ distinct from each other and from all recorders and predictors.

We define $\mathcal{Y}$ to be the space of all sequences of length ω of symbols. Here we return to normal set-theoretic convention by considering the domain of such sequences to be $\omega = \{0, 1, 2, \ldots\}$.

On $\mathcal{Y}$ we may define the shift function, which we again denote by $\mathfrak{s}$: $\mathfrak{s}(\zeta)(n) = \zeta(n+1)$ for $n \geqslant 0$.

As in section 4 of *Delays* we write $\zeta \triangleright \xi$, read ζ *is near to* ξ, if $\zeta = \mathfrak{s}^n(\xi)$ for some $n \geqslant 0$.

4·0 DEFINITION The *weight* of a point ζ of $\mathcal{Y}$ is the supremum of the weight of its predictors: thus either a natural number or ∞. The *height* of a point $\zeta \in \mathcal{Y}$ is the supremum of the height of its recorders and predictors: again either a natural number or ∞.

Introducing the real b

At last we are in a position to define our point b, which will lie in the space $\mathcal{Y}$.

4·1 DEFINITION Enumerate all sequences z_s^u where $u \in \mathcal{F}$ and s is a u-sequence, in some recursive fashion as z_i $(i = 0, 1, \ldots)$.

Define

$$b =_{\text{df}} z_0 {}^\frown \langle [\mathsf{m}_0] \rangle {}^\frown z_1 {}^\frown \langle [\mathsf{m}_1] \rangle {}^\frown \ldots$$

4·2 THEOREM $\theta(b, \mathfrak{s}) = \omega_1$.

To classify the points of $\mathcal{Y}$ attacked by b, we shall use the infinite trees to which the members of $\mathcal{F}$ are codes of finite approximations.

That's all for this week, folks !