E Pluribus Unum

Emergence of MoPA

Model Theory, Proof Theory, Set Theory, Recursion Theory, Computational Complexity, Algebra, …

Ken McAloon-Ryniak

Pre-History

- Dedekind
- Peano
- Thought in 2^{nd} Order Terms, **N** as categorical
- Frege and First Order Logic
- Principia Mathematica, Zermelo, Zermelo-Fraenkel Set Theory – all first order theories
- Hilbert's Program and Proof Theory
- Presburger Arithmetic and QE (1929)
  - Tarski assigned it as a Master's degree project

Gödel's Theorems

- Completeness Theorem (1930)
  - Yields existence of non-standard models of PA but first-order PA was likely not yet formalized
- Incompleteness Theorems (1931)
  - Hilbert's Program: Von Neumann and others had proved consistency of fragments of arithmetic
  - The arithmetic of "PM and related systems"
    - Not PA – so can assume certain facts about number theory (e.g. Chinese Remainder Theorem?)
    - Complex history of 2^{nd} Incompleteness Theorem – historian's delight, Königsberg

  - Self-reference and diagonalization
    - Who knows about first formulation? Is alluded to by Kripke and others
    - Dawson's books?Jon Von Plato's books?
    - Arithmetization, as we know it, suggested by Von Neumann
    - Gaifman's remarks at Cornell *etc*.
    - Subject very much alive and well
      - Saeed Salehi MoPA talk
      - Smorynski's paper The Early History of Formal Diagonalisation

    - Rosser's Theorems

The plot thickens

- Tarski and undefinability of Truth (1933)
- Concept of Truth in Formalized Languages – birth of Model Theory
- Skolem’s construction of non-standard models (1934)
  - Here is (as far as I can tell) first formulation of first order PA
  - Like an ultra-power construction

Recursion Theory

- 1930's – the Golden Age
  - Herbrand-Gödel Recursive Functions, λ-calculus, Kleene's T-Predicate, ..., Turing Machines
  - Church's Thesis (now the Church Turing Thesis – fair enough because Gödel only began to believe the Thesis after seeing Turing's work)
  - Turing's PhD thesis and paper: *Systems of Logic Based on Ordinals*
  - Kleene: "The Germans had this Proof Theory and we were trying to catch up."

Proof Theory

- Hilbert-Bernay's book (1938)
  - Hilbert Derivability Conditions, proof of 2nd Theorem
- Gentzen's work (1930s, early 1940s)
  - Brilliant, original
    - Faithful to Hilbert's program
    - Göttingen – graduate student of Bernays, also worked with Hilbert and Weyl
  - Poorly understood "in my day"
    - A consistency proof for PA – Quixotic
    - Assigned ε_0 to PA – cool but vague
    - His work and Schütte's - all in German
    - Tait, Spector, Curry, Howard, *et al*.

Après-Guerre

- Leon Henkin
  - Elegant proof of the Completeness Theorem
  - Henkin's Problem, Löb's Theorem
  - Arithmetic Completeness Theorem
  - Order type of < in countable non-standard models
    - $\omega + (\omega^* + \omega)^*\eta$
- Tennenbaum (1959) no non-standard model with recursive operations
- Non-finite Axiomatizability of PA
  - Ryll-Nardzewski (1952)
    - Proof uses non-standard models

Fin de l'Apres-Guerre

- *Infinitistic Methods* (Meeting in Warsaw 1959)
    - MacDowell-Specker Theorem – a classic – a countable model of PA model has an elementary end extension
        - Timeless: used by Jim Schmerl in 2006 paper on minimal end extensions
    - Richard Montague – reflection and finite axiomatizabillity
    - Andrej Mostowski – non-finite axiomatizability of theories
    - Dana Scott – constructing models is much more difficult for PA than Geometry because of quantifier changes in axioms and undecidability – all this makes it difficult to show things are independent or whatever by building non-standard models.

- Feferman (FM 1960) *Arithmetization of Metamathematics in a General Setting*

Hierarchies of Recursive Functions

- Reportedly Gödel said that a hierarchy for the Recursive Functions was an important open problem.
- Church and his students (*e.g.* Joel Robbin) worked on hierarchies of sub-recursive families of functions (1950s, 60s)
  - Gossip: there was someone who wrote his thesis under Church but switched fields immediately after - telling Ralph Abraham that the field was boring
  - But history will more than validate this work when MoPA starts to catch up with Proof Theory
- The Grzegorczyk hierarchy (1953)

1960s - Stalking Hilbert's Tenth

- Rabin
  - Non-standard models and the independence of the induction axiom (1961)
  - Models of Arithmetic and Diophantine Equations (1962)
- Davis, Davis Putnam Robinson
  - Almost there – need for equation whose solutions exhibit exponential growth
- In 1970, Matisasevitch's Theorem (MDRP)

1970s – The Confluence I

- Haim Gaifman
  - Work on types was a big step in serious elegant Model Theory - 1970 paper
  - Key short paper on MDRP and MoPA: $\Sigma 1$ = E1
  - Key long paper: *Models and types of Peano's arithmetic*. Annals of Mathematical Logic, vol. 9(1976), pp. 223–306.
    - Gems: minimal end extension theorem
- Joram Hirshfeld's thesis: *Existentially Complete and Generic Models for Arithmetic*
  - Student of Abraham Robinson
- Julia Knight
  - Papers on Omitting Types and Hanf Numbers, student of Tarski
- Alex Wilkie (The Open University) several papers among them
  - *On Models of Arithmetic – Answers to Two Problems Raised by H. Gaifman*
- Friedman's Isomorphism Theorem SLN 337 (1973)

The Confluence II

- Angus MacIntyre-Harry Simmons
  - *Gödel's diagonalization technique and related properties of theories*
  - "precursor to the modal logic treatment of provability and diagonalization" (Craig Smorynski) – e.g. Guaspari-Solovay (1979), von Bülow
  - Harry Simmons' talk in Paris: exploiting recursion theory to build models, thus giving a new life to those results
- Abramson-Harrington
  - *Models without indiscernibles (1976)*
    - Nice combinatorics: Nesteril—Rödl Theorem
- McAloon TAMS *Completeness Theorems, Incompleteness Theorems and Models of Arithmetic (1978)* – e.g. MD requires all PA
- Ulf Schmerl's extension of Löb's Theorem – most useful for working with ordinal notations
- Reverse Mathematics (Friedman 1975) Subsystems of 2nd Order Arithmetic
  - Major program: Simpson *et al.*

The Confluence III

- Parikh (1971) Existence and Feasibility in Arithmetic
  - Esenine-Volpin’s “hyper-finitism” in background
  - Applied by Paris later to solve a Solovay problem
  - Cegielski paper
- Grigori Mints: The provably recursive funtions of Σ_1^0 induction are the primitive recursive functions.
  - In Russian, Luc Pirdeni to the rescue

The Confluence IV

- Complexity Theory, e.g. Ferrante,Rackoff on Presburger Arithmetic
- Presburger Arithmetic applied to Automated Reasoning (1970s already)
- Wainer (1972) "working in the fields of the Lord" at Leeds on what is now known as the *Wainer Hierarchy*
- Girard thought assigning ordinals like ε_0 to PA was "stupide"
  - So he developed a system of ordinal notations and a slow hierarchy that took Γ_0 (the Feferman-Schütte ordinal) steps to outrun the provably recursive functions of PA
    - Notations used dilators – hard to follow

Set Theory and Manchester

- Work of Paris and Kirby and Kirby-Paris
  - Think of cuts in non-standard models of PA as large cardinals (weakly compact) – strong cuts *etc*.
  - Indicators expose richness of initial segments of such models
    - Very British – games where each move is a "go"
- Jeff's first true, unprovable combinatorial statement
  - A (finite) set X is 0-dense if card(X) >= min X + 3
  - A set X is (n+ 1)-dense if for every 2-coloring of the 3 element subsets of X, there is an n-dense homogeneous subset
  - Thm: For all n, there exist n-dense sets
    - Uses MacDowell-Specker (which requires all of PA)
  - n-dense captures set theory's "infinity"
  - It provides an indicator which yields strong cuts where the theorem itself must fail
  - Equivalent to 1-Consistency : Con(PA + Π_1^0 - truth)

Paris-Harrington and All That

- (Wikipedia) For any positive integers n, k, m, such that $m \geq n$, one can find N with the following property: if we color each of the n-element subsets of $S = \{1, 2, 3, \ldots, N\}$ with one of k colors, then we can find a subset Y of S with at least m elements, such that all n-element subsets of Y have the same color, and the number of elements of Y is at least the smallest element of Y.
  - Equivalent to 1-Consistency of PA
  - Quickly published in the volume *Handbook of Mathematical Logic* (ed. Barwise, 1977)

Onward

- Kirby-Paris (1982)
  - Hydra
    - Proof by animation
  - Goodstein's Theorem
    - ε_0 was explicitly needed for the original proof
  - A great British moment
    - OBE level

Meanwhile Back in Paris

- Gaifman's course on types *etc*. (1977)
- *Modèles de l'arithmétique de Peano*, Astérisque, 73, Société Mathématique de France
  - Even got Model Theory stalwarts Jean-Pierre Ressayre, Max Dickmann and Daniel Lascar into the mix
- Action Thématique Programmée (ATP) (1979-80)
  - *Model Theory and Arithmetic* (LNM, 890) (1981)
  - Peter Clote, Patrick Cegielski, Zoé Chatzidakis, Anand Pillay
  - Pascal Michel, Denis Richard, Peter Aczel
  - Cherlin-Dickmann, Paris, Wilmers, Wilkie, McAloon-Ressayre
- Kirby-Murawski-McAloon paper (1979)
- Zofia Adamowicz

Further Onward

- Ketonen-Solovay (Annals 1981) Rapidly Growing Ramsey Functions
  - Wainer hierarchy to do Paris-Harrington etc.
- Friedman-McAloon-Simpson
  - *A Finite Combinatorial Principle Which is Equivalent to the 1-Consistency of Predicative Analysis (1982)*
  - A combinatorial statement Poincaré would not have been able to prove
  - Shamelessly invoked Γ_0
- Friedman and Reverse Mathematics: a version of Kruskal's Theorem is not provable in ATR_0
  - "early 1980s" according to Wikipedia
- Kanamori-McAloon (1987)
  - Started with notes on large cardinals by Ketonen
  - Regressive functions came from set theory
  - No need for "large" finite sets

Recursion Theory Revisited

- Smorynski's papers on MoPA and recursive saturation (1980s)
- Direct recursion-theoretic proof of Kirby-Paris-Goodstein result by E.A.Chicon (1983)
- Harrington's solution of McAloon's problem:
  - Construct a model of PA with arithmetic operations but non-arithmetic truth set
    - Dazzling – Chaim and I were trying to decipher it in Paris, I remember
    - Dave Marker - first person to really get it
- Clote-McAloon *Yet Two More …*
  - Based on anti-basis theorems in Clote's paper in ATP
  - analogous to Jockusch/Ramsey's Theorem/Paris-Harrington

Spill Out to Other Fields

- Wilkie’s proof of Gromov’s Theorem
  - Groups of polynomial growth
    - Finite by nil-potent
    - Non-standard algebra
  - Meeting at Brooklyn College
    - Kirby, Mate, Wilkie, Yours Truly, …

Spill Out, cont

- Complexity Theory
  - KM: Finite Reachable Petri Nets (Containment problem is primitive recursive in the Ackermann function, uses large finite sets).
  - KM and Mike Anshel: decision problems for HNN groups
  - Jeff Paris' notes
  - Clote and others
  - Ajtai's (muscular) paper with finite Borel sets
    - *Sigma$_1^1$ Formulae on Finite Structures*
    - Isomorphic integers of different parities in different models
    - New proof of Furst-Sipser-SaxeTheorem on parity and circuits of bounded depth:

  A super-polynomial lower bound is given for the size of circuits of fixed depth computing the parity function.

Subsystems of PA

- Ehrenfeucht-Jensen (1976 FM)
- Cegielski
  - Multiplication paper in ATP volume
  - Paper with McAloon and Wilmers on Recursive Saturation
  - Kept the faith: Journées sur les Arithmétiques Faibles (JAF)
- Kaye, Paris, Dimitracopoulos
  - *On parameter free induction schemas*, by Kaye, R. W., Paris, J.B., and Dimitracopoulos, C. *The Journal of Symbolic Logic* 53 (1988) 1082--97.
  - Dimitracopoulos too kept the faith: Journées sur les Arithmétiques Faibles (JAF)
- Buss – bad ass proof theory
- Kirby
  - “functional” formulations

A Place in the Sun

- Nice combinatorial proofs of things like the equivalence of Paris-Harrington and Kanamori-McAloon results
- Andreas Weiermann and others – elegant fine analysis of fast functions
- Macho model theory – Kossak, Schmerl, Lascar and others
- Book by Hajek and Pudlak
  - *Metamathematics of First-Order Arithmetic* (1993)
- Book by Kossak and Schmerl
  - *The Structure of Models of Peano Arithmetic* (2006)
- Book by Richard Kaye
  - *Models of Peano arithmetic* (1991)
- Kotlarski's posthumous work
  - *A model-theoretic approach to proof theory* (2019)
- The CUNY MoPA zoominar
- Journées sur les l'Arithmétiques Faibles

Patrick Cégielski's
Email to 161 people

- Subject: Official cancellation of JAF (Journées sur les Arithmétiques Faibles) 2022 in Moscow (June,13-17)

- Dear all,
- *We wish to inform you that JAF 41 was officially cancelled by the Steering Committee following the United Nations' overwhelming resolution concerning the Russian invasion of Ukraine.*
-
- Of course we have to recognize the hard work of our Russian colleagues to organize an issue which looked promising.
-
- Best regards,
-
- Patrick

Athens Poster

- https://conferences.uoa.gr/event/30/images/117-JAF40_Poster.png

FMS independent statement

- Let X be a finite set of positive integers. A *coloring* of X is given by a partition P(X) = C_1 U C_2 where C_1 and C_2 are closed under initial segment. A subset Y of X is *homogeneous* if either P(Y) is included in C1 or P(Y) is included in C2.
- The finite set X is said to be *0-dense* if card(X) >= 2 and card(X) >= min X; X is *n+1 dense* if every coloring of X has an n-dense homogeneous subset.
- Theorem: For all n, there exist an n-dense finite set.

Kanamori-McAloon

- For all n,k in N there exists a m such that for any regressive function f on the k element subsets of {1,…,m} there is a subset H with at least n elements such that for any k element subset S of H the value of f(S) only depends on min S.

- [Ga70] Haim Gaifman, On local arithmetical functions and their application for constructing types of Peano's arithmetic in: Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968) North-Holland, Amsterdam, 1970, pp. 105–121.
- [Kn76] Julia F. Knight, Omitting types in set theory and arithmetic, J. Symbolic Logic 41 (1976), 25–32.

Hierarchies of Recursive Functions, revisited

- Ketonen-Solovay (1981)
  - A direct proof of the Paris-Harrington Theorem
  - Rate of growth of recursive functions
  - Based on Wainer hierarchy
  - *Annals of Mathematics* paper
  - Realized a conjecture of Peter Aczel (in the ATP volume) on the role of ε_0