Initial self-embeddings of models of set theory

joint work with Ali Enayat (University of Gothenburg)

Zach McKenzie

H. Friedman's Self-embedding Theorems

H. Friedman, *Countable models of set theory* (1973):

Theorem

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be countable nonstandard models of power admissible set theory that also satisfy separation for formulae that are Σ in the expanded language that includes the unary powerset operation. Then $\mathcal{M}$ is isomorphic to an initial segment of $\mathcal{N}$ that is a union of ranks (V_αs) of $\mathcal{N}$ if and only if

(i) *$\mathcal{M}$ and $\mathcal{N}$ have the same well-founded sets and code the same subclasses of these well-founded sets, AND*

(ii) *for any Σ-formula in the expanded language that contains the unary powerset operation, $\phi(\vec{a})$, with parameters that are well-founded sets, if $\mathcal{M}$ satisfies $\phi(\vec{a})$ then so does $\mathcal{N}$.*

Theorem

Every countable nonstandard model of ZF is isomorphic to a proper initial segment consisting of a union of ranks (V_αs) of itself.

H. Friedman's Self-embedding Theorems

The arguments used to prove these results also work in the context of arithmetic:

Theorem

Every countable nonstandard model of PA is isomorphic to a proper initial segment of itself.

In both versions of H. Friedman's Theorem (for set theory and arithmetic):

- The images of the embeddings can be made to be Σ_n-elementary substructures (for any fixed concrete n)
- The embedding can be made to pointwise fix any proper initial segment of the model

In the context of set theory, H. Friedman's theorem guarantees the existence of an embedding whose image is initial in the sense that its image is a union of ranks.

Preliminaries

There also two weaker notions of an initial segment of a model of set theory:

- Transitive subclass
- Transitive subclass that contains all subsets

In this talk we will investigate versions of H. Friedman's Self-embedding Theorem that guarantee the existence of an embedding whose image is a transitive subclass. This will also allow us to examine self-embeddings of models of set theory that do not satisfy the powerset axiom.

Throughout this talk I will use $\mathcal{L}$ to denote the language of set theory– first-order logic with a binary membership relation ($\in$).

- In addition to the Lévy classes of formulae $\Delta_0, \Sigma_1, \Pi_1, \ldots$ we will also have cause to consider the Takahashi classes $\Delta_0^{\mathcal{P}}, \Sigma_1^{\mathcal{P}}, \Pi_1^{\mathcal{P}}, \ldots$

Definition

The class $\Delta_0^{\mathcal{P}}$ is the class of formulae built inductively from atomic formulae using the connectives $\wedge$, $\vee$, $\neg$ and $\Rightarrow$, and quantification in the form $\mathcal{Q}x \in y$ and $\mathcal{Q}x \subseteq y$ where $\mathcal{Q}$ is either $\exists$ or $\forall$, and x and y are distinct free variables.

Preliminaries

- $\Sigma_1^{\mathcal{P}}$ is the class of formulae in the form $\exists \vec{x} \phi$ where ϕ is $\Delta_0^{\mathcal{P}}$
- $\Pi_1^{\mathcal{P}}$ is the class of formulae in the form $\forall \vec{x} \phi$ where ϕ is $\Delta_0^{\mathcal{P}}$
- etc.

Let Γ be a class of $\mathcal{L}$-formulae.

(Γ-separation) For all $\phi(x, \vec{z}) \in \Gamma$,

$$\forall \vec{z} \forall w \exists y \forall x (x \in y \iff (x \in w) \wedge \phi(x, \vec{z})).$$

(Γ-collection) For all $\phi(x, y, \vec{z}) \in \Gamma$,

$$\forall \vec{z} \forall w ((\forall x \in w) \exists y \phi(x, y, \vec{z}) \Rightarrow \exists C (\forall x \in w)(\exists y \in C) \phi(x, y, \vec{z})).$$

(Γ-foundation) For all $\phi(x, \vec{z}) \in \Gamma$,

$$\forall \vec{z} (\exists x \phi(x, \vec{z}) \Rightarrow \exists y (\phi(y, \vec{z}) \wedge (\forall x \in y) \neg \phi(x, \vec{z}))).$$

If $\Gamma = \{x \in z\}$ then we will refer to Γ-foundation as **set foundation**.

Preliminaries

(TCo) Every set is contained in a transitive set.

(WO) Every set can be well-ordered.

(Axiom H) For every cardinal κ, there exists a transitive set T that contains every transitive set with cardinality $\leq \kappa$ (this says that we have sets that look like the H_κs).

The α-Dependent Choice Scheme is the natural class version of dependent choice:

($\Pi^1_\infty - \mathrm{DC}_\alpha$) For all $\mathcal{L}$-formulae $\phi(x, y, \vec{z})$,

$$\forall \vec{z} \left(\begin{array}{c} \forall g (\forall \gamma \in \alpha) \left(\begin{array}{c} (g \text{ is a function}) \wedge (\mathrm{dom}(g) = \gamma) \Rightarrow \\ \exists y \phi(g, y, \vec{z}) \end{array} \right) \Rightarrow \\ \exists f \left(\begin{array}{c} (f \text{ is a function}) \wedge (\mathrm{dom}(f) = \alpha) \\ \wedge (\forall \beta \in \alpha) \phi(f \restriction \beta, f(\beta), \vec{z}) \end{array} \right) \end{array} \right).$$

Note that in the absence of powerset, Zermelo's Well-ordering Principle (WO) is not equivalent to the Axiom of Choice (Zarach (1982)).

Kripke-Platek Set Theory (with infinity)

Definition

KPI *is the $\mathcal{L}$-theory with axioms: extensionality, emptyset, pair, union, infinity, Δ_0-separation, Δ_0-collection and Π_1-foundation.*

This is the modern presentation of Kripke-Platek Set Theory with the Axiom of Infinity. Unlike the Admissible Set Theory studied in H. Friedman's 1973 paper and the Kripke-Platek Set Theory of Barwise's "Admissible Sets and Structure", KPI only includes Π_1-foundation instead of full class foundation. A transitive set that is either V_ω or a model of KPI is called an **admissible set**.

- KPI is capable of defining the rank function (ρ) and this definition is Δ_1.
- KPI is capable of defining satisfaction in set structures.
- KPI is capable of defining the levels of L (the L_αs) and proving that the function $\alpha \mapsto L_\alpha$ is Δ_1.
- KPI is capable of defining the partial satisfaction predicates $\mathrm{Sat}_{\Sigma_n}(m, x)$ and $\mathrm{Sat}_{\Pi_n}(m, x)$ and proving that these predicates are Σ_n and Π_n respectively.

$\mathrm{KP}^{\mathcal{P}}$

Definition

$\mathrm{KP}^{\mathcal{P}}$ is obtained from KPI by adding powerset, $\Delta_0^{\mathcal{P}}$-collection and $\Pi_1^{\mathcal{P}}$-foundation.

This is the modern presentation of H. Friedman's Power Admissible Set Theory from his 1973 paper. Again, $\mathrm{KP}^{\mathcal{P}}$ only includes $\Pi_1^{\mathcal{P}}$-foundation instead of full class foundation.

- $\mathrm{KP}^{\mathcal{P}}$ proves the $\Sigma_1^{\mathcal{P}}$-Recursion Theorem.
- $\mathrm{KP}^{\mathcal{P}}$ proves that for all ordinals α, V_α exists, and that the function $\alpha \mapsto V_\alpha$ is $\Delta_1^{\mathcal{P}}$.
- $\mathrm{KP}^{\mathcal{P}}$ does not prove that there is no largest cardinal (in fact there is a transitive model of $\mathrm{KP}^{\mathcal{P}}$ with only recursive ordinals).
- $\mathrm{KP}^{\mathcal{P}}$ does not prove Axiom H. In fact, Mathias (2001) shows that $\mathrm{KP}^{\mathcal{P}}$ plus the assertion that there is no largest cardinal does not prove Axiom H.

Mostowski Set Theory

Definition

MOST *is obtained from* KPI *by adding powerset,* Σ_1*-separation and* WO*.*

- Mathias (2001) has shown that MOST is weak (equiconsistent with Mac Lane Set Theory) and consistent with $V = L$.
- MOST proves that every well-founded extensional relation is isomorphic to a transitive set (Mostowski's Lemma).
- MOST proves that for every cardinal κ, H_κ exists (i.e. Axiom H holds).
- MOST is axiomatised by: extensionality, pair, union, powerset, TCo, infinity, Δ_0-separation, WO, set foundation and Axiom H.
- MOST does not prove that for every ordinal α, V_α exists.
- MOST does not prove that $\aleph_\omega$ exists, or even the statement $(\forall n \in \omega)(\aleph_n \text{ exists})$.

ZFC minus Powerset with Dependent Choices

Definition

Let ZFC *be Zermelo-Fraenkel Set Theory axiomatised using the collection scheme instead of the replacement scheme and* WO *instead of Choice.* ZFC^- *is obtained from* ZFC *by removing the powerset axiom.*
ZF^- *is obtained from* ZFC^- *by removing* WO*.*
$\mathrm{ZFC}^- + \forall\alpha(\Pi^1_\infty - \mathrm{DC}_\alpha)$ *is obtained from* ZFC^- *by adding, for all ordinals* α*, the scheme* $\Pi^1_\infty - \mathrm{DC}_\alpha$*.*

- S. Friedman, Gitman and Kanovei (2019) have recently shown that ZFC^- does not prove $\Pi^1_\infty - \mathrm{DC}_\omega$.
- Flanagan (1975) shows that a global well-order that can be used in all instances of the separation and collection scheme can be conservatively added to $\mathrm{ZFC}^- + \forall\alpha(\Pi^1_\infty - \mathrm{DC}_\alpha)$.

Three notions of initial self-embedding

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be $\mathcal{L}$-structures.

- We write $\mathcal{M} \subseteq_e \mathcal{N}$ if $\mathcal{N}$ is and end-extension of $\mathcal{M}$. I.e. $\mathcal{M}$ is a transitive subclass of $\mathcal{N}$.
- We write $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$ if $\mathcal{M} \subseteq_e \mathcal{N}$ and for all $x \in M$ and for all $y \in N$, if $\mathcal{N} \models (y \subseteq x)$, then $y \in M$.
- Let $\mathcal{M}, \mathcal{N} \models \mathrm{KPI}$. We say that $\mathcal{N}$ is a **rank extension** $\mathcal{M}$ if $\mathcal{M} \subseteq_e \mathcal{N}$ and for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ and for all $x \in N$, if $\mathcal{N} \models (\alpha = \rho(x))$, then $x \in M$.

Example

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a model of ZFC. Then

- $\langle \omega_3^{\mathcal{M}}, \in^{\mathcal{M}} \rangle \subseteq_e \mathcal{M}$
- $\langle H_{\aleph_3}^{\mathcal{M}}, \in^{\mathcal{M}} \rangle \subseteq_e^{\mathcal{P}} \mathcal{M}$
- *$\mathcal{M}$ is a rank extension of $\langle V_{\omega_3}^{\mathcal{M}}, \in^{\mathcal{M}} \rangle$*

Three notions of initial self-embedding

Let $\mathcal{M} \models \mathrm{KPI}$ and let $j : \mathcal{M} \longrightarrow \mathcal{M}$ be an embedding (an injective function that preserves $\in^{\mathcal{M}}$). Write $j[\mathcal{M}]$ for the substructure of $\mathcal{M}$ corresponding to the image of j.

- We say that j is a **proper initial self-embedding** if $j[\mathcal{M}] \subseteq_e \mathcal{M}$ and $j[\mathcal{M}] \neq \mathcal{M}$.
- We say that j is a **proper $\mathcal{P}$-initial self-embedding** if $j[\mathcal{M}] \subseteq_e^{\mathcal{P}} \mathcal{M}$ and $j[\mathcal{M}] \neq \mathcal{M}$.
- We say that j is a **proper rank-initial self-embedding** if $\mathcal{M}$ is a rank extension of $j[\mathcal{M}]$ and $j[\mathcal{M}] \neq \mathcal{M}$.

Theorem

(Gorbow (2018), implicit in H. Friedman (1973)) Let $\mathcal{M} \models \mathrm{KP}^{\mathcal{P}}$. If j is a proper $\mathcal{P}$-initial self-embedding, then j is a proper rank-initial self-embedding.

Self-embedding Theorems

Theorem

(Gorbow (2018)) Every countable nonstandard model $\mathcal{M}$ of $\mathrm{KP}^{\mathcal{P}} + \Sigma_1^{\mathcal{P}}$-separation has a proper rank-initial self-embedding. Moreover, given any $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, there exists a proper rank-initial self-embedding of $\mathcal{M}$ that fixes every element of $V_\alpha^{\mathcal{M}}$.

Theorem

(Enayat, Kaufmann, M. (2018)) Every countable recursively saturated model of $\mathrm{MOST} + \Pi_1$-collection admits a proper $\mathcal{P}$-initial self-embedding.

The well-founded part

Definition

*Let $\mathcal{M} \models \mathrm{KPI}$. The **well-founded part** or **standard part** of $\mathcal{M}$, denoted $\mathrm{WF}(\mathcal{M})$, is the substructure of $\mathcal{M}$ with underlying set, $\mathrm{WF}(M)$, that consists of all sets x such that $\in^{\mathcal{M}}$ is well-founded on $\mathrm{TC}(\{x\})$. If $\mathrm{WF}(\mathcal{M}) \neq \mathcal{M}$, then we say that $\mathcal{M}$ is* **nonstandard**. *The **standard ordinals** of $\mathcal{M}$, denoted $\mathrm{o}(\mathcal{M})$, is the substructure of $\mathcal{M}$ with underlying set $\mathrm{o}(M) = \mathrm{WF}(M) \cap \mathrm{Ord}^{\mathcal{M}}$. If $\omega^{\mathcal{M}} \in \mathrm{o}(M)$, then we say that $\mathcal{M}$ is **ω-standard**. Mostowski's Collapsing Lemma ensures that both $\mathrm{o}(\mathcal{M})$ and $\mathrm{WF}(\mathcal{M})$ are isomorphic to transitive sets. In particular, $\mathrm{o}(\mathcal{M})$ is isomorphic to an ordinal that is called the **standard ordinal** of $\mathcal{M}$.*

Lemma

If $\mathcal{M}$ is a nonstandard model of KPI, then $\mathrm{WF}(\mathcal{M}) \subseteq_e^{\mathcal{P}} \mathcal{M}$.

Theorem

(H. Friedman (1973), Barwise (1975)) If $\mathcal{M}$ is a nonstandard model of $\mathrm{KPI} + \Sigma_1\textit{-foundation}$, then $\mathrm{WF}(\mathcal{M})$ is admissible.

Initial self-embeddings and the well-founded part

Lemma

Let $\mathcal{M} \models \mathrm{KPI}$. If $j : \mathcal{M} \longrightarrow \mathcal{M}$ is an initial self-embedding, then j is the identity on $\mathrm{WF}(\mathcal{M})$.

Proof.

External $\in$-induction. □

Lemma

Let $\mathcal{M} \models \mathrm{KPI}$ and let $j : \mathcal{M} \longrightarrow \mathcal{M}$ be an initial self-embedding. If $x \in \mathcal{M}$ is definable Σ_1-formula with parameters that are fixed by j, then x is fixed by j.

Proof.

Suppose that x is the unique element of $\mathcal{M}$ such that $\mathcal{M} \models \phi(x, \vec{a})$, where ϕ is Σ_1 and $\vec{a}$ are fixed by j. So, since $\vec{a}$ is fixed, $j[\mathcal{M}] \models \phi(j(x), \vec{a})$. Since $j[\mathcal{M}] \subseteq_e \mathcal{M}$ and ϕ is Σ_1, $\mathcal{M} \models \phi(j(x), \vec{a})$, which implies that x is fixed. □

A nonstandard model of ZFC^- that has no initial self-embedding

Definition

Let $\mathcal{M} \models \mathrm{KPI}$. (a) The well-founded part of $\mathcal{M}$ is ***c-bounded*** *in $\mathcal{M}$, where "c" stands for "cardinalitywise", if there is some $x \in M$ such that for all $w \in \mathrm{WF}(M)$ $\mathcal{M} \models |x| > |w|$. (b) The well-founded part of $\mathcal{M}$ is* ***c-unbounded*** *in $\mathcal{M}$ if the well-founded part of $\mathcal{M}$ is not c-bounded in $\mathcal{M}$, i.e., if for all $x \in M$, there exists $w \in \mathrm{WF}(M)$ such that $\mathcal{M} \models |x| \leq |w|$.*

Theorem

Let $\mathcal{M} \models \mathrm{KPI}$. If the well-founded part of $\mathcal{M}$ is **c-unbounded**, *then $\mathcal{M}$ admits no proper initial self-embedding.*

Proof.

Let $x \in M$. Let $w \in \mathrm{WF}(M)$ be such that $\mathcal{M} \models (|x| \leq |w|)$. There is a topped well-founded extensional relation $R_x \subseteq w \times w \in \mathrm{WF}(M)$ that is isomorphic to $\mathrm{TC}(\{x\})$. This yields of Σ_1-formula with parameters w and R_x that defines x. □

A nonstandard model of ZFC^- that has no initial self-embedding

Theorem

There is a countable nonstandard model of ZFC^- that admits no proper initial self-embedding.

Proof.

Let $\mathcal{M}$ be a model of ZFC with standard ω, but nonstandard ω_1. Then $\langle H_{\aleph_1}^{\mathcal{M}}, \in^{\mathcal{M}} \rangle$ is a model of ZFC^- such that the well-founded part of this model is c-unbounded. □

Uncountable non-standard models of ZFC with no initial self-embeddings

Definition

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a model of ZF. We say that $\mathcal{M}$ is $\aleph_1$-like if $|M| = \aleph_1$, but for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, $\{x \in M \mid \mathcal{M} \models (x \in V_\alpha)\}$ is countable.

Theorem

(Keisler and Morely (1968)) Every consistent extension of ZF has an $\aleph_1$-like model.

It is clear that any $\aleph_1$-like model of ZF will not admit a proper rank-initial self-embedding.

Uncountable non-standard models of ZFC with no initial self-embeddings

Theorem

If $\mathcal{M}$ is an $\aleph_1$-like model of $\mathrm{ZF} + V = L$, then $\mathcal{M}$ admits no proper initial self-embedding.

Proof.

Suppose that $j : \mathcal{M} \longrightarrow \mathcal{M}$ is an embedding such that $j[\mathcal{M}] \subseteq_e \mathcal{M}$. The fact that $\mathcal{M}$ is $\aleph_1$-like ensures that $j[\mathrm{Ord}^{\mathcal{M}}] = \mathrm{Ord}^{\mathcal{M}}$. And, using $V = L$, every element of $\mathcal{M}$ is definable by a Σ_1-formula with ordinal parameters. This implies that $j[\mathcal{M}] = \mathcal{M}$. □

Summary

- We have introduced the notion of "initial self-embedding" that is more general than the "rank-initial self-embeddings" studied by H. Friedman.
- We have shown that if $\mathcal{M}$ is a model of KPI such that every set in $\mathcal{M}$ is the same size as a well-founded set (the well-founded part of $\mathcal{M}$ is c-unbounded), then $\mathcal{M}$ admits no initial self-embedding.
- This allows us to construct a nonstandard model of ZFC^- that is not isomorphic to a transitive subclass of itself.
- Next time, we will investigate when proper initial self-embedding do exist.

Thank you!

Brief recap. . .

Let $\mathcal{M} \models \mathrm{KPI}$ and let $j : \mathcal{M} \longrightarrow \mathcal{M}$ be an embedding (an injective function that preserves $\in^{\mathcal{M}}$). Write $j[\mathcal{M}]$ for the substructure of $\mathcal{M}$ corresponding to the image of j.

- We say that j is a **proper initial self-embedding** if $j[\mathcal{M}] \subseteq_e \mathcal{M}$ and $j[\mathcal{M}] \neq \mathcal{M}$.
- We say that j is a **proper $\mathcal{P}$-initial self-embedding** if $j[\mathcal{M}] \subseteq_e^{\mathcal{P}} \mathcal{M}$ and $j[\mathcal{M}] \neq \mathcal{M}$.
- We say that j is a **proper rank-initial self-embedding** if $\mathcal{M}$ is a rank extension of $j[\mathcal{M}]$ and $j[\mathcal{M}] \neq \mathcal{M}$.

In the last talk we showed that if $\mathcal{M} \models \mathrm{KPI}$ and every set in $\mathcal{M}$ is (according to $\mathcal{M}$) the same size as well-founded set (the well-founded part of $\mathcal{M}$ is c-unbounded), then $\mathcal{M}$ admits no proper initial self-embedding. This allowed us to show that there is a nonstandard model of ZFC^- with no proper initial self-embedding. Today we investigate when nonstandard models of KPI do admit proper initial self-embeddings. . .

Constructing initial self-embeddings

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$. If $a \in M$, then we use a^* to denote the set $\{x \in M \mid \mathcal{M} \models (x \in a)\}$.

Definition

Let $\mathcal{M} \models \mathrm{KPI}$. The **standard system** *of $\mathcal{M}$ is the set*

$$\mathrm{SSy}(\mathcal{M}) = \{y^* \cap \mathrm{WF}(M) \mid y \in M\}$$

If $A \in \mathrm{SSy}(\mathcal{M})$ and $y \in M$ is such that $y^* \cap \mathrm{WF}(\mathcal{M}) = A$, then we say y **codes** *A.*

Definition

Let $\mathcal{M} \models \mathrm{KPI}$. We say that the well-founded part of $\mathcal{M}$ is **contained** *if there exists $c \in M$ such that $\mathrm{WF}(M) \subseteq c^*$.*

Lemma

Let $\mathcal{M} \models \mathrm{KPI}$. If $\mathcal{M}$ is ω-nonstandard, then the well-founded part of $\mathcal{M}$ is contained.

Constructing initial self-embeddings

Proof.
$\mathrm{WF}(M) \subseteq (L_\omega^{\mathcal{M}})^*$. □

Lemma
Let $n \in \omega$ and let $m = \max\{1, n\}$. Let $\mathcal{M} \models \mathrm{KPI} + \Sigma_m$-separation. If the well-founded part of $\mathcal{M}$ is contained and $\vec{a} \in M$, then

$$\{\langle \ulcorner \phi(x, \vec{y}) \urcorner, b\rangle \mid \phi \textit{ is } \Sigma_n, b \in \mathrm{WF}(M) \textit{ and } \mathcal{M} \models \phi(b, \vec{a})\} \in \mathrm{SSy}(\mathcal{M})$$

Proof.
Let $\vec{a} \in M$ and c be such that $\mathrm{WF}(M) \subseteq c^*$. Since $\mathrm{Sat}_{\Sigma_n}(m, x)$ is Σ_m, we can use Σ_m-separation to find y with $\mathcal{M} \models (y \subseteq c)$ that codes this class. □

When our model is ω-nonstandard, we replace the use of separation above by a overspill argument.

Constructing initial self-embeddings

Lemma

Let $n \in \omega$ and let $m = \max\{1, n\}$. Let $\mathcal{M} \models \mathrm{KPI} + \Pi_{m-1}$-collection $+ \Pi_{n+1}$-foundation be such that $\mathcal{M}$ is ω-nonstandard. If $\vec{a} \in M$, then

$$\{\langle \ulcorner \phi(x, \vec{y}) \urcorner, b \rangle \mid \phi \textit{ is } \Sigma_n, b \in \mathrm{WF}(M) \textit{ and } \mathcal{M} \models \phi(b, \vec{a})\} \in \mathrm{SSy}(\mathcal{M})$$

Proof.

Note that

$$A = \{\langle \ulcorner \phi(x, \vec{y}) \urcorner, b \rangle \mid \phi \text{ is } \Sigma_n, b \in \mathrm{WF}(M) \text{ and } \mathcal{M} \models \phi(b, \vec{a})\}$$

is a subclass of V_ω. For all $n \in \omega$, there is a set $y_n \in M$ such that y_n codes $A \cap V_n$. We can "overspill" this inside $\mathcal{M}$ to get $y \in M$ that codes A. □

Constructing initial self-embeddings

Theorem

Let $p \in \omega$ and let $\mathcal{M}$ be a countable model of $\mathrm{KPI} + \Sigma_{p+1}$-separation $+ \Pi_p$-collection. Let $b, B \in M$ and $c \in B^*$ with the following properties:

(I) $\mathcal{M} \models \bigcup B \subseteq B$

(II) $\mathrm{WF}(M) \subseteq B^*$

(III) *for all Π_p-formulae $\phi(\vec{x}, y, z)$ and for all $a \in \mathrm{WF}(M)$,*

$$\textit{if } \mathcal{M} \models \exists \vec{x} \phi(\vec{x}, a, b), \textit{ then } \mathcal{M} \models (\exists \vec{x} \in B)\phi(\vec{x}, a, c).$$

Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq B^*$, $j(b) = c$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Proof.

(Sketch) A proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ is obtained by using a back-and-forth argument to construct sequences $\langle u_i \mid i \in \omega \rangle$ and $\langle v_i \mid i \in \omega \rangle$, and defining $j(u_i) = v_i$ for all $i \in \omega$.

Constructing initial self-embeddings

Proof.

(Sketch, continued.) After stage $n \in \omega$, $u_0, \ldots, u_n \in M$ and $v_0, \ldots, v_n \in B^*$ will have been chosen so as to maintain:

> ($\dagger_n$) for all Π_p-formulae, $\phi(\vec{x}, z, y_0, \ldots, y_n)$, and for all $a \in \mathrm{WF}(M)$, if $\mathcal{M} \models \exists\vec{x}\phi(\vec{x}, a, u_0, \ldots, u_n)$, then $\mathcal{M} \models (\exists\vec{x} \in B)\phi(\vec{x}, a, v_0, \ldots, v_n)$.

The "forth" stage of the construction chooses u_n in order to ensure that the domain of j is all of M. The coding lemmas on the preceding slides and ($\dagger_{n-1}$) allow a corresponding $v_n \in B^*$ to be chosen to maintain ($\dagger_n$). The "back" stage of the construction eventually ensures that the image of the embedding is transitive. If v_n is in the transitive closure of $\{v_0, \ldots, v_{n-1}\}$, then the coding lemmas on the preceding slides and ($\dagger_{n-1}$) are used to choose u_n to maintain ($\dagger_n$). □

Constructing initial self-embeddings

The same back-and-forth argument with "separation" replaced by "overspill" when coding classes yields:

Theorem

Let $p \in \omega$ and let $\mathcal{M}$ be a countable model of $\mathrm{KPI} + \Pi_p\text{-}\mathit{collection} + \Pi_{p+2}\text{-}\mathit{foundation}$ that is ω-nonstandard. Let $b, B \in M$ and $c \in B^*$ with the following properties:

(I) $\mathcal{M} \models \bigcup B \subseteq B$

(II) $\mathrm{WF}(M) \subseteq B^*$

(III) *for all Π_p-formulae $\phi(\vec{x}, y)$,*

$$\textit{if } \mathcal{M} \models \exists \vec{x} \phi(\vec{x}, b), \textit{ then } \mathcal{M} \models (\exists \vec{x} \in B)\phi(\vec{x}, c).$$

Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq B^*$, $j(b) = c$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Constructing initial self-embeddings

Theorem

Let $p \in \omega$, $\mathcal{M}$ be a countable model of $\mathrm{KPI} + \Sigma_{p+1}\textit{-separation} + \Pi_p\textit{-collection}$ such that the well-founded part of $\mathcal{M}$ is contained, and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $b \in \mathrm{rng}(j)$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Proof.

$\Sigma_{p+1}\text{-separation} + \Pi_p\text{-collection}$ $(= \text{strong } \Pi_p\text{-collection})$ can be used to obtain a transitive set B that satisfies the conditions called for in the preceding self-embedding results from a set C that contains the well-founded part of $\mathcal{M}$. □

Theorem

Let $p \in \omega$, $\mathcal{M}$ be a countable ω-nonstandard model of $\mathrm{KPI} + \Pi_p\textit{-collection} + \Pi_{p+2}\textit{-foundation}$, and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $b \in \mathrm{rng}(j)$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Nonstandard models with proper initial self-embeddings

Corollary

Let $\mathcal{M} \models \mathrm{KPI} + \Sigma_1\text{-}separation$ be such that the well-founded part of $\mathcal{M}$ is contained. Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$.

Corollary

Let $\mathcal{M} \models \mathrm{KPI} + \Pi_2\text{-}foundation$ be ω-nonstandard. Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$.

Question

Is every ω-nonstandard model of KPI isomorphic to a proper transitive subclass of itself?

Question

If $\mathcal{M} \models \mathrm{KPI}$ is such that the well-founded part of $\mathcal{M}$ is contained, then does $\mathcal{M}$ admit a proper initial self-embedding?

A model of $\mathrm{MOST} + \Pi_1\text{-collection}$ with a proper initial self-embedding but no proper $\mathcal{P}$-initial self-embedding

- Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a countable ω-nonstandard model of $\mathrm{ZFC} + V = L$.
- Define a substructure $\mathcal{N} = \langle N, \in^{\mathcal{M}} \rangle$ by

$$N = \bigcup_{n \in \omega} (H_{\aleph_n}^{\mathcal{M}})^* \text{ where } (H_{\aleph_n}^{\mathcal{M}})^* = \{x \in M \mid \mathcal{M} \models (x \in H_{\aleph_n})\}$$

- $\mathcal{N}$ is an ω-nonstandard model of $\mathrm{MOST} + \Pi_1\text{-collection}$. Therefore $\mathcal{N}$ admits a proper initial self-embedding.
- The infinite cardinals of $\mathcal{N}$ are exactly $\aleph_n$ where n is a standard natural number. Therefore, since any $\mathcal{P}$-initial embedding preserves cardinals, $\mathcal{N}$ admits no proper $\mathcal{P}$-initial embedding.

A model of $\mathrm{MOST} + \Pi_1\text{-collection}$ with a proper $\mathcal{P}$-initial self-embedding but no proper rank-initial self-embedding

- The model $\mathcal{N}$ outlined on the previous slide satisfies

  (†) For all cardinals κ, there exists a set X with cardinality κ and countable rank.

  To see this consider $V_\omega, \mathcal{P}(V_\omega), \mathcal{P}^2(V_\omega), \ldots$.
- Let $\mathcal{Q}$ be a countable recursively saturated model of $\mathrm{MOST} + \Pi_1\text{-collection} + (\dagger)$. Therefore $\mathcal{Q}$ has a proper $\mathcal{P}$-initial self-embedding.
- If $h : \mathcal{Q} \longrightarrow \mathcal{Q}$ is a proper $\mathcal{P}$-initial self-embedding, then h must map H_κs from $\mathcal{Q}$ to $H_{h(\kappa)}$ in $\mathcal{Q}$ (everything is $H_{h(\kappa)}$ is coded as a subset of $h(\kappa)$). But this means that there must be a cardinal in $\mathcal{Q}$ that is not in the image of h and therefore a set of countable rank that is also not included in the image of h.

Every countable ω-nonstandard model of ZF is isomorphic to a transitive subclass of HC of its own L

The Lévy-Shoenfield Absoluteness Theorem shows that if $\mathcal{M}$ is a model of ZF, then $\mathcal{M}$ and $H_{\aleph_1}^{L^{\mathcal{M}}}$ have satisfy the same Σ_1 theory. This yields the following embedding result:

Theorem

Let $\mathcal{M}$ be a countable model of ZF that is not ω-standard. Then there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq_e (H_{\kappa}^{L^{\mathcal{M}}})^*$, where $\kappa = (\aleph_1^L)^{\mathcal{M}}$.

Proof.

Work inside $\mathcal{M}$. Let $B = H_{\aleph_1}^L$. Note that B is transitive, $\mathrm{WF}(M) \subseteq B^*$ and, by the Lévy-Shoenfield Absoluteness Theorem, for all Δ_0-formulae, $\phi(\vec{x}, z)$, if $\mathcal{M} \models \exists \vec{x} \phi(\vec{x}, \emptyset)$, then $\mathcal{M} \models (\exists \vec{x} \in B)\phi(\vec{x}, \emptyset)$. Therefore, there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq B^* = (H_{\aleph_1}^{L^{\mathcal{M}}})^*$. □

Every countable ω-nonstandard model of ZF is isomorphic to a transitive subclass of HC of its own L

This is related to the following two results:

Theorem

(Barwise (1971)) Let $\mathcal{M}$ be a countable model of ZF. Then there exists structures $\mathcal{N}_1$ and $\mathcal{N}_2$ such that

(I) $\mathcal{M} \subseteq_e \mathcal{N}_1 \subseteq_e \mathcal{N}_2$,

(II) $\mathcal{N}_2 \models \mathrm{ZF} + V = L$, *and*

(III) $\mathcal{N}_1 = \langle (H_{\aleph_1}^{\mathcal{N}_2})^*, \in^{\mathcal{N}_2} \rangle$.

Theorem

(Hamkins (2013)) Let $\mathcal{M}$ be a countable model of ZF. Then there exists an embedding of $\mathcal{M}$ into its own L.

The relationship between 'contained' and 'c-unbounded'

Recall from last week:

Definition

Let $\mathcal{M} \models \mathrm{KPI}$. (a) The well-founded part of $\mathcal{M}$ is ***c-bounded*** *in $\mathcal{M}$, where "c" stands for "cardinalitywise", if there is some $x \in M$ such that for all $w \in \mathrm{WF}(M)$ $\mathcal{M} \models |x| > |w|$. (b) The well-founded part of $\mathcal{M}$ is* ***c-unbounded*** *in $\mathcal{M}$ if the well-founded part of $\mathcal{M}$ is not c-bounded in $\mathcal{M}$, i.e., if for all $x \in M$, there exists $w \in \mathrm{WF}(M)$ such that $\mathcal{M} \models |x| \leq |w|$.*

Theorem

Let $\mathcal{M} \models \mathrm{KPI}$. If the well-founded part of $\mathcal{M}$ is c-unbounded, then $\mathcal{M}$ admits no proper initial self-embedding.

We have seen that if $\mathcal{M}$ satisfies enough set theory and the well-founded part of $\mathcal{M}$ is contained, then $\mathcal{M}$ admits a proper initial self-embedding. So, what is the relationship between the well-founded part being contained and c-(un)bounded?

The relationship between ‘contained’ and ‘c-unbounded’

Lemma

Let $\mathcal{M} \models \mathrm{KPI}$ *be* ω*-standard. If the well-founded part of* $\mathcal{M}$ *is contained, then* $\mathrm{WF}(\mathcal{M}) \models \mathrm{KP}^{\mathcal{P}}$*.*

Proof.

Let $c \in M$ be such that $\mathrm{WF}(M) \subseteq c^*$. If $x \in \mathrm{WF}(M)$, then define $P(x) = \{y \in c \mid y \subseteq x\}$- this is a set by Δ_0-separation. The fact that $\mathrm{WF}(\mathcal{M}) \subseteq_e^{\mathcal{P}} \mathcal{M}$ ensures that $P(x)$ is the powerset of x in both $\mathcal{M}$ and $\mathrm{WF}(\mathcal{M})$, and implies that $\mathrm{WF}(\mathcal{M})$ satisfies Δ_0-separation and powerset. The fact that there is no least ordinal in $M \backslash \mathrm{WF}(M)$ ensures that $\mathrm{WF}(\mathcal{M})$ satisfies $\Delta_0^{\mathcal{P}}$-collection. □

Lemma

Let $\mathcal{M} \models \mathrm{KPI}$*. If the well-founded part of* $\mathcal{M}$ *is contained, then the well-founded part of* $\mathcal{M}$ *is c-bounded.*

The relationship between 'contained' and 'c-unbounded'

Proof.

Let $c \in M$ be such that $\mathrm{WF}(M) \subseteq c^*$. Since $\mathrm{WF}(M)$ is closed under the powerset operation from $\mathcal{M}$ implies that for all $w \in \mathrm{WF}(M)$, $\mathcal{M} \models |w| < |c|$. □

Lemma

Let $\mathcal{M}$ be a model of $\mathrm{ZFC}^- + \forall\alpha\ \Pi^1_\infty{-}\mathrm{DC}_\alpha$. If the well-founded part of $\mathcal{M}$ is c-bounded, then $\mathrm{WF}(\mathcal{M}) \models \mathrm{KP}^{\mathcal{P}}$.

Proof.

Let κ be an $\mathcal{M}$-cardinal such that for all $w \in \mathrm{WF}(\mathcal{M})$, $\mathcal{M} \models (|w| < \kappa)$. Note that it is enough to show that $\mathrm{WF}(\mathcal{M})$ satisfies powerset. Let $Y \in \mathrm{WF}(M)$ be such that Y has no powerset in $\mathrm{WF}(\mathcal{M})$. Consider sequences that enumerate the subsets of Y in $\mathcal{M}$. Any such sequence whose range is not the powerset of Y can be extended. Therefore, $\Pi^1_\infty{-}\mathrm{DC}_\kappa$ can be used to obtain a sequence of subsets of Y of cardinality $\geq \kappa$, which is a contradiction. □

The relationship between 'contained' and 'c-unbounded'

Lemma

Let $\mathcal{M}$ be a model of $\mathrm{ZFC}^- + \forall\alpha\ \Pi^1_\infty{-}\mathrm{DC}_\alpha$ such that the well-founded part of $\mathcal{M}$ is c-bounded. Then the well-founded part of $\mathcal{M}$ is contained.

Proof.

Since $\mathrm{WF}(\mathcal{M}) \models \mathrm{KP}^{\mathcal{P}}$, for all $\alpha \in o(\mathcal{M})$, $\mathcal{M} \models (V_\alpha \text{ exists})$. Moreover, this will 'overspill' to give us a V_β in $\mathcal{M}$, where β is ill-founded, and this set will contain the well-founded part of $\mathcal{M}$. □

Theorem

Let $\mathcal{M}$ be a countable nonstandard model of $\mathrm{ZFC}^- + \forall\alpha\ \Pi^1_\infty{-}\mathrm{DC}_\alpha$. Then the following are equivalent:

(I) *The well-founded part of $\mathcal{M}$ is c-bounded,*

(II) $\mathrm{WF}(\mathcal{M}) \models \mathrm{KP}^{\mathcal{P}}$,

(III) *For all $n \in \omega$ and for all $b \in M$, there exists a proper initial self-embedding $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that $b \in \mathrm{rng}(j)$ and $j[\mathcal{M}] \prec_n \mathcal{M}$.*

Elementary submodels of models of ZF^-

The result on the previous slide is related to the following:

Theorem

(Quinsey (1980)) Let $n \in \omega$. If $\mathcal{M} \models \mathrm{ZF}^-$ is nonstandard, then there exists $\mathcal{N} \subseteq_e \mathcal{M}$ such that $\mathcal{N} \neq \mathcal{M}$, $\mathcal{N} \prec_n \mathcal{M}$ and $\mathcal{N} \models \mathrm{ZF}^-$.

We have shown that if $\mathcal{M}$ is countable and satisfies Dependent Choices in the above, then there exists $\mathcal{N} \prec_n \mathcal{M}$ such that $\mathcal{N} \cong \mathcal{M}$, $\mathcal{N} \subseteq_e \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$ exactly when the well-founded part of $\mathcal{M}$ is c-bounded.

Question

Is there a countable model $\mathcal{M}$ of ZFC^- such that the well-founded part of $\mathcal{M}$ is c-bounded in $\mathcal{M}$, but $\mathcal{M}$ does not admit a proper initial self-embedding?

Thank you!