The Kaufmann–Clote question on end extensions of models of arithmetic and the weak regularity principle

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Fragments of first- and second-order arithemetic

- The language of first-/ second-order arithmetic:
  $\mathcal{L}_1 = \{+, \times, <, =, 0, 1\}$, $\mathcal{L}_2 = \{+, \times, <, =, 0, 1, \in\}$.
- $\Delta^0_0, \Sigma^0_n, \Pi^0_n$-formulas are defined by counting the number of blocks of unbounded quantifiers.
- A formula is Δ^0_n if it is equivalent to both a Σ^0_n and a Π^0_n formula (over some model or theory).
- $\mathrm{I}\Sigma^0_n$ consists of PA^- and **Induction** for all Σ^0_n formulas φ:

  $$\varphi(0, \overline{c}) \wedge (\forall x\ (\varphi(x, \overline{c}) \to \varphi(x+1, \overline{c})) \to \forall x\ \varphi(x, \overline{c}).$$

- $\mathrm{B}\Sigma^0_n$ consists of $\mathrm{I}\Delta^0_0$ and **Collection** for all Σ^0_n formulas φ:

  $$\forall x < a\ \exists y\ \varphi(x, y, \overline{c}) \to \exists b\ \forall x < a\ \exists y < b\ \varphi(x, y, \overline{c}).$$

- (Paris–Kirby 1978) $\mathrm{I}\Delta^0_0 + \exp \dashv \mathrm{B}\Sigma^0_1 + \exp \dashv \mathrm{I}\Sigma^0_1 \dashv \mathrm{B}\Sigma^0_2 \dashv \mathrm{I}\Sigma^0_2 \ldots$ and none of the converses holds.
- $\mathrm{RCA}_0 = \mathrm{I}\Sigma^0_1 + \Delta^0_1$-comprehension.
  $\mathrm{WKL}_0 = \mathrm{RCA}_0 +$"each infinite binary tree has an infinite path".

End extensions

Definition

Let $M, K \models \mathrm{PA}^-$, $M \subseteq K$ is an **end extension** if

$$\forall x \in K \setminus M \; \forall y \in M \; y < x.$$

Denote it by $M \subseteq_{\mathrm{e}} K$. We say that the extension is **proper** if $K \neq M$, and it is n-**elementary** if all the Σ_n-formulas are absolute between M and K.

End extensions vs Elementarity

End extensions with elementarity provides a model-theoretic characterization of the strength of induction/collection in the ground model.

Theorem (MacDowell–Specker 1959)

Every model of PA *has a fully elementary proper end extension.*

Theorem (Paris–Kirby 1978)

For any $n \in \mathbb{N}$*, let* $M \models \mathrm{I}\Delta_0 + \exp$ *be countable, then*

$$\exists K \neq M,\ M \preccurlyeq_{\mathrm{e},n+2} K \iff M \models \mathrm{B}\Sigma_{n+2}.$$

In particular, if M *has a fully elementary proper end extension, then* $M \models \mathrm{PA}$*, i.e., the converse of MacDowell–Specker Theorem holds.*

Generalizing Paris–Kirby

Theorem (Paris–Kirby 1978)

For any $n \in \mathbb{N}$, *let* $M \models \mathrm{I}\Delta_0 + \exp$ *be countable, then*

$$\exists K \neq M,\ M \preccurlyeq_{\mathrm{e},n+2} K \iff M \models \mathrm{B}\Sigma_{n+2}.$$

Questions

- What end extension property characterizes $M \models \mathrm{I}\Sigma_{n+2}$?
- For which theory T, we can always let $K \models T$ in the Paris–Kirby Theorem?

$\mathrm{I}\Sigma_n$	✓	trivial from $(n+2)$-elementarity
$\mathrm{B}\Sigma_{n+1}$	?	
$\mathrm{I}\Sigma_{n+1}$	X	implies $M \models \mathrm{B}\Sigma_{n+3}$

The Kaufmann–Clote question

This remaining question is due to Clote, where he mentioned that the same question is raised by Kaufmann in the context of models of set theory.

Question (Kaufmann–Clote)

Let $n \in \mathbb{N}$, does every countable model of $\mathrm{B}\Sigma_{n+2}$ admit a proper $(n+2)$-elementary end extension $K \models \mathrm{B}\Sigma_{n+1}$?

Theorem (S.)

Yes.

Another story: the regularity principle

Definition (Regularity principle)

- Let $\varphi(x, y)$ be a first-order formula. Then $\mathrm{R}\varphi$ denotes the universal closure of the following formula:

$$\exists^{\mathrm{cf}} x \; \exists y < a \; \varphi(x, y) \rightarrow \exists y < a \; \exists^{\mathrm{cf}} x \; \varphi(x, y).$$

- For any formula class Γ, $\mathrm{R}\Gamma = \mathrm{I}\Delta_0 + \{\mathrm{R}\varphi \mid \varphi \in \Gamma\}$.

Theorem (Mills–Paris 1984)

For each $n \in \mathbb{N}$, $\mathrm{B}\Sigma_{n+2} \Leftrightarrow \mathrm{R}\Sigma_{n+1} \Leftrightarrow \mathrm{R}\Pi_n$.

Regularity principle vs End extensions

The existence of proper end extension with elementarity indicates certain regularity principle via a 'nonstandard analysis' argument.

Proposition

Let $M \models \mathrm{I}\Delta_0 + \exp$. If $\exists K \neq M$, $M \preccurlyeq_{\mathrm{e},n+2} K$, then $M \models \mathrm{R}\Pi_n$.

Proof.

Let $M \models \exists^{\mathrm{cf}} x \ \exists y < a \ \varphi(x, y)$ for some $\varphi(x, y) \in \Pi_n$.

1. Transfer this Π_{n+2}-statement to K by elementarity.
2. Pick some $d > M$ in K such that $K \models \varphi(d, c)$ for some $c < a$. Then for any $b \in M$, $K \models \exists x > b \ \varphi(x, c)$.
3. Transfer each of these statements back to M. Then $M \models \exists^{\mathrm{cf}} x \ \varphi(x, c)$.

□

Regularity principle vs End extensions

Proposition

Let $M \models \mathrm{I}\Delta_0 + \exp$. If $\exists K \neq M$, $M \preccurlyeq_{\mathrm{e},n+2} K \models \mathrm{B}\Sigma_{n+1}$, then

$$M \models \forall x\ \exists y < a\ \varphi(x,y) \rightarrow \exists y < a\ \exists^{\mathrm{cf}} x\ \varphi(x,y)$$

for any $\varphi \in \Pi_{n+1}$.

Proof.

Same as the previous proof. Notice that in step 1, $\forall x\ \exists y < a\ \varphi(x,y)$ is Π_{n+2} over $\mathrm{B}\Sigma_{n+1}$, so this statement correctly transfers to K. □

We call this formula the **weak regularity principle** $\mathrm{WR}\varphi$.

Corollary

If K–C question has a positive answer, then $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$.

A syntactic proof of $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$ via WKL_0

Proposition

$\mathrm{B}\Sigma^0_2 + \mathrm{WKL}_0 \vdash \mathrm{WR}\Pi^0_1$.

Proof.

Let $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_2 + \mathrm{WKL}_0 + \forall x\ \exists y < a\ \forall z\ \theta(x, y, z)$ for some $\theta \in \Delta^0_0$. Consider the following a-branching tree T:

$$\sigma \in T \iff \forall x, z < \mathrm{len}\,\sigma\ \theta(x, \sigma(x), z).$$

By $\mathrm{I}\Sigma^0_1$, for each $l \in M$ there is a σ with $\mathrm{len}\,\sigma = l$ such that $\forall z\ \theta(x, \sigma(x), z)$, so T is infinite. Pick a infinite path $P \in \mathcal{X}$ of T. Then $(M, \mathcal{X}) \models \forall x\ \forall z\ \theta(x, P(x), z)$. Finally, pick a $c < a$ such that $\exists^{\mathrm{cf}} x\ P(x) = c$ by $\mathrm{B}\Sigma^0_2$. □

The first-order version follows by a standard relativization argument and the fact that WKL_0 is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$.

Question

How does such argument relate to the K–C question?

From first- to second-order ultrapower

- Paris–Kirby's construction is based on a (first-order) Δ_{n+1}-ultrapower construction.
- One can show that such Δ_{n+1}-ultrapowers always fail to satisfy $\mathrm{B}\Sigma_{n+1}$ in the K–C question.
- The syntactic proof above indicates that we need to work in a second-order context with WKL_0.

From first- to second-order ultrapower

Definition (Second-order ultrapower)

Let $(M, \mathcal{X}) \models \mathrm{I}\Sigma^0_1$, and $\mathcal{U}$ be an ultrafilter on $\mathcal{X}$ such that all the $A \in \mathcal{U}$ are cofinal in M. Then the **second-order ultrapower** $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ is defined to be $(\mathcal{F}/\sim, \mathcal{X})$, where

- $\mathcal{F}$ is the class of total functions in $\mathcal{X}$ and $f \sim g \iff \{x \in M \mid f(x) = g(x)\} \in \mathcal{U}$.
- For any $A \in \mathcal{X}$, $[f] \in A \iff \{x \in M \mid f(x) \in A\} \in \mathcal{U}$.

Theorem (Łoś, essentially Kirby 1984)

- *For any* $\varphi(x) \in \Sigma^0_1(M, \mathcal{X})$,

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \varphi([f]) \iff \exists A \in \mathcal{X}, \{x \in M \mid (M, \mathcal{X}) \models \varphi(f(x))\} \supseteq A \in \mathcal{U}.$$

- $(M, \mathcal{X}) \preccurlyeq_{\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$.

The construction

By a relativization argument and expansion to WKL_0, it suffice to show the following:

Lemma

For any countable $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_2 + \mathrm{WKL}_0$*, there is a second-order ultrapower* $(M, \mathcal{X}) \subseteq_{\mathrm{e}} (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$*.*

Proof.

The argument for '$\subseteq_{\mathrm{e}}$' is standard: For each $f \in \mathcal{F}$, if $[f] < a \in M$ is forced to be true, that is

$$\{x \in M \mid f(x) < a\} \in \mathcal{U}$$

then by $\mathrm{B}\Sigma^0_2$, we can let $\{x \in M \mid f(x) = b\} \in \mathcal{U}$ for some $b < a$, and thus $[f] = b$. Otherwise, we can always force $[f] > a$ by setting

$$\{x \in M \mid f(x) > a\} \in \mathcal{U}.$$

The construction (cont.)

To have $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$, we need to 'force' all the instances of $\mathrm{B}\Sigma^0_1$ step by step:

$$\forall y < [g] \; \exists z \; \theta([f], y, z) \to \exists b \; \forall y < [g] \; \exists z < b \; \theta([f], y, z),$$

where $\theta \in \Delta^0_0$ and f, g are total functions in $\mathcal{X}$.

Strategy: We always try to force the conclusion to be true in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$. That is, try to set

$$\{x \in M \mid \exists d \; \forall y < g(x) \; \exists z < d \; \theta(f(x), y, z)\} \supseteq A \in \mathcal{U}$$

for some $A \in \mathcal{X}$. If we succeed, then by Łoś's theorem, we are done. If not, then we can make use of the extra information given by the failure.

The construction (cont.)

The failure means $\{x \in M \mid \exists d\ \forall y < g(x)\ \exists z < d\ \theta(f(x), y, z)\}$ is bounded. By $\mathrm{B}\Sigma_1^0$, there is a $b \in M$ such that:

$$(M, \mathcal{X}) \models \forall x > b\ \exists y < g(x)\ \forall z\ \neg\theta(f(x), y, z).$$

Similar to the proof of $\mathrm{B}\Sigma_2^0 + \mathrm{WKL}_0 \vdash \mathrm{WR}\Pi_1^0$, we can construct a finite branching tree T, and there is a infinite path(total function) $P \in \mathcal{F}$ of T bounded by g, such that

$$(M, \mathcal{X}) \models \forall x > b\ \forall z\ \neg\theta(x, P(x), z).$$

Such $P \in \mathcal{F}$ provides a witness of $\exists y < [g]\ \forall z\ \neg\theta([f], y, z)$ by Łoś's theorem. □

Theorem

For any $n \in \mathbb{N}$ and countable $M \models \mathrm{B}\Sigma_{n+2}$, there is a $(n+2)$-elementary proper end extension $M \subseteq_{\mathrm{e}} K \models \mathrm{B}\Sigma_{n+1}$.

The strength of $\mathrm{WR}\Gamma$ in the I-B hierarchy

We continue analyzing the strength of $\mathrm{WR}\Gamma$.

Theorem

For each $n \in \mathbb{N}$,

- $\mathrm{B}\Sigma_{n+2} \Leftrightarrow \mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1}) \Leftrightarrow \mathrm{WR}\Sigma_0(\Sigma_n)$.
- $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}) \vdash \mathrm{I}\Sigma_{n+2}$.

Question

For which formula class Γ, $\mathrm{WR}\Gamma \Leftrightarrow \mathrm{I}\Sigma_{n+2}$?
Does $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}) \Leftrightarrow \mathrm{I}\Sigma_{n+2}$?
(Which end extension property characterize $M \models \mathrm{I}\Sigma_{n+2}$?)

Induction up to an initial segment

Proposition

For each $n \in \mathbb{N}$, let M, K be models of $\mathrm{I}\Delta_0 + \exp$ and $M \subseteq_{\mathrm{e}} K$. Then TFAE:

- For any $\varphi(x, y) \in \Sigma_{n+1}(K)$ and $a \in M$,

$$K \models \exists b \; \forall x < a \; (\exists y \; \varphi(x, y) \leftrightarrow \exists y < b \; \varphi(x, y)).$$

- For any $\varphi(x) \in \Sigma_{n+1}(K)$ and $a \in M$, $\{x < a \mid K \models \varphi(x)\}$ is coded in K (and actually in M).

We call them $K \models M\text{-}\mathrm{I}\Sigma_{n+1}$.

Regularity principle vs End extension (cont.)

There is also a corresponding 'nonstandard analysis' argument for $K \models M\text{-}\mathrm{I}\Sigma_{n+1}$.

Proposition

Let $M \models \mathrm{I}\Delta_0 + \exp$. If $\exists K \neq M$, $M \preccurlyeq_{\mathrm{e},n+2} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$, then $M \models \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$.

Proof.

The proof is still the same. Notice that $\forall x\ \exists y < a\ (\varphi(x,y) \wedge \psi(x,y))$ is equivalent to a Π_{n+2}-formula over $\mathrm{I}\Sigma_{n+1}$, and actually over $M\text{-}\mathrm{I}\Sigma_{n+1}$ since $a \in M$. So again this statement correctly transfers to K. □

Remark

Actually the argument above proves $M \models \mathrm{WR}\varphi$, where $\varphi(x,y) \in \Sigma_0(\Sigma_{n+1})$, and x does not appear in the bound of a bounded quantifier. In particular, $M \models \mathrm{WR}\varphi$ if $\varphi(x,y)$ is a Boolean combination of Σ_{n+1}-formulas.

Characterizing $\mathrm{I}\Sigma_{n+2}$ by end extensions

Proposition

For any $n \in \mathbb{N}$ and countable $M \models \mathrm{I}\Sigma_{n+2}$,
$\exists K \neq M$, $M \preccurlyeq_{\mathrm{e},n+2} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$.

Again, it suffice to show the second-order version. This time, we don't need WKL_0.

Proposition (Second-order version)

For any countable $(M, \mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$, there is a second-order ultrapower $(M, \mathcal{X}) \subseteq_{\mathrm{e}} (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models M\text{-}\mathrm{I}\Sigma^0_1$.

The proof is mild generalization of Clote (1985), where he proves that every countable $M \models \mathrm{I}\Sigma_{n+2}$ has a proper $(n+2)$-elementary end extension to some $K \models M\text{-}\mathrm{B}\Sigma_{n+1}$, which is defined similar to $M\text{-}\mathrm{I}\Sigma_{n+1}$.

The construction

Proof.

For each uniform sequence of Σ^0_1-definable sets $\{A_i\}_{i<b}$, say $A_i = \{x \in M \mid \varphi(f(x), i)\}$ where $\varphi \in \Sigma^0_1$ and $f \in \mathcal{F}$, we try to maximize

$$\{i < b \mid \exists A \in \mathcal{X}\ , A_i \supseteq A \in \mathcal{U}\}.$$

That is, let $B \in \mathcal{X}$ be the intersection of all the subsets currently enumerated into $\mathcal{U}$. Take the largest $c < 2^b$ such that

$$\bigcap_{i \in \mathrm{Ack}(c)} A_i \cap B \text{ is cofinal in } M,$$

and put a subset of this set in $\mathcal{X}$ into $\mathcal{U}$. Then such c will code $\{i < b \mid (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \varphi([f], i)\}$. □

Characterizing $\mathrm{I}\Sigma_{n+2}$ by end extensions

Theorem (S.)

For any $n \in \mathbb{N}$, *let* $M \models \mathrm{I}\Delta_0 + \exp$ *be countable, then*

$$\exists K \neq M,\ M \preccurlyeq_{\mathrm{e},n+2} K \models M\text{-}\mathrm{I}\Sigma_{n+1} \iff M \models \mathrm{I}\Sigma_{n+2}.$$

Theorem (S.)

For each $n \in \mathbb{N}$, $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}) \Leftrightarrow \mathrm{I}\Sigma_{n+2}$.

Question

Is there a syntactic proof of the equivalence above?

Summary

- K–C question has a positive answer:

$$\exists K \neq M,\ M \preccurlyeq_{\mathrm{e},n+2} K \models \mathrm{B}\Sigma_{n+1} \iff M \models \mathrm{B}\Sigma_{n+2}.$$

- A model-theoretic characterization of $\mathrm{I}\Sigma_{n+2}$:

$$\exists K \neq M,\ M \preccurlyeq_{\mathrm{e},n+2} K \models M\text{-}\mathrm{I}\Sigma_{n+1} \iff M \models \mathrm{I}\Sigma_{n+2}.$$

- The strength of the weak regularity principle:

$$\mathrm{B}\Sigma_{n+2} \Leftrightarrow \mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1}) \Leftrightarrow \mathrm{WR}\Sigma_0(\Sigma_n).$$

$$\mathrm{I}\Sigma_{n+2} \Leftrightarrow \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}).$$

Thank You!