Choiceless probability paradoxes via box games

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Abstract
The purpose of these notes is to prove and discuss some surprising theorems of Zermelo set theory without choice (Z). These theorems are arguably as counterintuitive as various so-called "paradoxes" of ZFC and other strong set theories, including the Banach-Tarski paradox, winning strategies for the infinite prisoners puzzle, and the failure of Freiling's Axiom of Symmetry in ZFC+CH. Such paradoxes have been used to argue against certain axioms used in their proofs, particularly the Axiom of Choice (AC). We will argue that these reductio ad ridiculum arguments against set-theoretic axioms are not epistemically sound by observing that these arguments rest on intuitions that fail in less controversial theories like Z and even third-order arithmetic (Z_3).

Arguments against choice and CH

We begin by discussing several arguments against the axiom of choice of the form "the axiom of choice can be used to derive [some counterintuitive conclusion], so the axiom of choice must be false." Perhaps the origin of such arguments is Stefan Banach's and Alfred Tarski's work on paradoxical decompositions, the most famous being a decomposition of a solid ball into five pieces which can be rearranged via isometries into two solid balls, each of the same radius as the original.[1] This result, now known as the Banach-Tarski paradox, is a theorem of ZFC (or merely ZF plus the existence of a well-ordering of the real numbers), and it implies that there is no total finitely additive probability measure on the unit cube invariant under isometry. This can be considered a paradox of probability, since naively one expects that we can assign a probability to any event of the form "a randomly chosen element of the unit cube is in a fixed subset S," and that these probabilities should be at the very least finitely additive.

Now let's consider Chris Freiling's argument against the Continuum Hypothesis (CH), taking for granted the veracity of ZFC. Consider the following axiom: "For every function $f : [0,1] \to \mathcal{P}_{\omega_1}([0,1])$, there exists $x, y \in [0,1]$ such that $x \notin f(y)$ and $y \notin f(x)$." This is now known as Freiling's Axiom of Symmetry (AX), and it is equivalent to $\neg CH$. For our purposes, it suffices to show that AX is inconsistent with CH. Working in

ZFC+CH, there exists a well-ordering $<_w$ of the real numbers of length ω_1. Let $f : x \mapsto \{y \in [0,1] : y \leq_w x\}$. It is clear that there is no pair $x, y \in [0,1]$ such that $x \notin f(y)$ and $y \notin f(x)$, so AX fails.

Freiling argues that this is paradoxical, since a randomly chosen element pair $(x, y) \in [0,1]^2$ should satisfy $y \notin f(x)$, no matter which x is chosen (since $f(x)$ has measure 0), and symmetrically, it should satisfy $x \notin f(y)$. Therefore, he argues that CH is false.[2] However, others argue that this is really an argument against the existence of any well-ordering of $\mathbb{R}$, since the intuition that countable sets of reals have measure 0 should also apply to any set of reals of cardinality less than the continuum, so the paradox occurs even under this weaker assumption.[4]

A more modern argument against choice comes from a mathematical puzzle that has spread across the internet over various math enthusiast websites, including Greg Muller's blog post "The Axiom of Choice is Wrong." [5] The puzzle is as follows: A countably infinite sequence of prisoners are sentenced to be executed, but the warden gives each one a chance to survive. After a night to strategize, the prisoners are to be lined up, with prisoner number 1 seeing all other prisoners, prisoner 2 seeing prisoner 3 and onwards, etc. Each prisoner will be given a hat colored by some real number. The prisoners will simultaneously guess their hat colors without any chance for communication, and anyone who guesses correctly will be spared. Show there is a strategy which guarantees only finitely many prisoners will die.

Here is a proof that such a strategy exists. Assuming AC, there exists a well-ordering of the real numbers. The prisoners will agree to a well-ordering $<_w$ of the real numbers (or equivalently, of the countable sequences of reals) during the previous night. When the warden gives the prisoners their hats, this will determine an infinite sequence of real numbers. Each prisoner sees the tail of this sequence (i.e., the sequence of reals modded out by finitely many terms). Every prisoner can determine the $<_w$-least sequence s of reals which has the same tail as the hat sequence, and prisoner number n guesses their hat color to be s_n. Since s differs from the hat sequence by only finitely many terms, only finitely many prisoners will die. Thus, from AC, we have shown that a seemingly impossible task has a winning strategy. Indeed, like the first two paradoxes, this result can be derived by merely assuming the existence of a well-ordering of the real numbers.

Finally, we consider one more well-known mathematical puzzle that seems impossible, but has a solution if there exists a well-ordering of the reals. 100 mathematicians are strategizing outside a room (they may not communicate once the game starts). In the room is a countably infinite sequence of boxes, each containing a real number. One at a time, the mathematicians will enter the room and start opening the boxes in arbitrary order, until he is ready to guess the contents of some unopened box. Then he will close the boxes and leave the room. The mathematicians win the game if at most one mathematician makes an incorrect guess about which real is contained in the box he selects. Show there is a strategy which guarantees the group succeeds. [6]

Here is a strategy that works given a well-ordering of the reals $<_w$. Split the boxes into 100 subsequences $\{s^i\}_0^{99}$, each s^i containing all boxes congruent to i mod 100. Mathematician n will open all boxes except the ones in s^n. For each $m \neq n$, he determines the $<_w$-least sequence t^m of reals which has the same tail has the sequence of reals in s^m, and then determines the least k such that for all $m \neq n$, s^m and t^m agree at the kth term and beyond. Finally, mathematician n opens all boxes in s_n except the kth box, determines the $<_w$-least sequence t^n of reals which has the same tail as the sequence of reals in s^n, and guesses that the remaining box has value t^n_k. The only mathematician who can fail is one whose subsequence of boxes takes longer than all the others to agree with the $<_w$-least real sequence to share its tail. Thus, despite the fact that each mathematician individually seems to have an infinitesimal chance of making a correct guess, if they employ a winning strategy, each has at least a 99% chance of being one of the mathematicians to succeed.[1]

We thus have four paradoxes of probability that can all be derived from a well-ordering of $\mathbb{R}$. Should the combined weight of these paradoxes be considered a sufficient argument that $\mathbb{R}$ cannot be well-ordered? As we will demonstrate in this paper, equally severe paradoxes can be derived from choiceless theories, including Z and even Z_3.

Note that the paradoxical nature of the above results depends on a commitment to a realist notion of a randomly chosen real number, or equivalently, the belief that performing an infinite sequence of truly random coin flips is a coherent notion. This note will examine the paradoxical results of performing uncountably many coin flips, without committing to AC.

[1]It's a good exercise to extend this result to countably many mathematicians. Using full ZFC, it can be shown there are strategies for the box game with κ boxes, κ mathematicians, and λ values, for any infinite κ, λ.

A first example

We will begin by discussing one striking example of a choiceless paradox. Consider the following game, a variant of the classical infinite boxes puzzle:

Game 1. *Countably many mathematicians are strategizing outside of a room (they may not communicate once the game starts). In the room is one labeled box for each set of reals, containing within it some real (not necessarily an element of its label). Each mathematician will separately enter the room and start opening sets of boxes in arbitrary order, until he is ready to guess the contents of some unopened box. The mathematicians win the game if at most one mathematician makes an incorrect guess about which real is contained in the box he selects. Find an explicit winning strategy for the group of mathematicians.*

This version of the game seems even harder than the original, since this time infinitely many mathematicians have to guess successfully as opposed to a finite number, with the failure condition being as strict as in the original. The only sense in which this puzzle is easier is that there are more boxes in the room, but these extra boxes shouldn't be of use to any individual mathematician, who still has to guess the contents of some box he knows nothing about. Yet, this version has an explicit strategy, which can be formalized in Z set theory, or even third-order arithmetic.[2]

We will make use of the fact that for any equivalence relation $\sim$ on $\mathbb{R}$ and any function $f : \mathcal{P}(\mathbb{R}) \to \mathbb{R}/\sim$, there is a canonical pair $X \neq Y$ such that $f(X) = f(Y)$. Recursively define $X_\alpha = \bigcup_{\beta<\alpha} f(X_\beta)$. This sequence stabilizes precisely at the least α such that $f(X_\alpha) = f(X_\beta)$ for some $\beta < \alpha$, so $X = X_\alpha$ and $Y = X_\beta$ are as desired. [3]

We will apply this trick to two equivalence relations. For $g_i : \omega \times \omega \to \mathbb{R}$, we say $g_1 \sim g_2$ if they disagree on only finitely many columns. For $h_i : \omega \to \mathbb{R}$, we say $h_1 \sim_0 h_2$ if they agree cofinitely often. Of course, both of these can be identified as equivalence relations on $\mathbb{R}$.

We now describe the strategy σ_n for the nth mathematician. Notice that we can identify $\mathcal{P}(\mathbb{R})$ with $\mathcal{P}(\mathbb{R}) \times \omega \times \omega$, and the information hidden

[2]To be precise, this result is formalized in Z_3P, third-order arithmetic with an additional unary predicate in the language and the comprehension scheme extended to include formulae involving this predicate. The predicate encodes the reals inside the boxes.

[3]This construction is performed in third-order arithmetic by quantifying over sets of reals which encode sequences of the form $\langle X_\beta \rangle_{\beta<\alpha}$. See [3] for more details on this sort of construction.

in the boxes by a function $F : \mathcal{P}(\mathbb{R}) \times \omega \times \omega \to \mathbb{R}$. Open all boxes in $\mathcal{P}(\mathbb{R}) \times (\omega \setminus \{n\}) \times \omega$.

We define a function $f : \mathcal{P}(\mathbb{R}) \to ({}^{\omega\times\omega}\mathbb{R})/\sim$ by $X \mapsto [(k,l) \mapsto F(X,k,l)]_\sim$. Notice that the nth mathematician can deduce f from the information he has already acquired. Let $X \neq Y$ be the canonical pair such that $f(X) = f(Y)$.

Let $S_1 = \{m \neq n : |\{l < \omega : F(X,m,l) \neq F(Y,m,l)\}| < \aleph_0\}$, and let $S_2 = \omega \setminus (S_1 \cup \{n\})$. Notice that S_2 is finite. For $m \in S_1$, let l_m be least such that for $l \geq l_m$, $F(X,m,l) = F(Y,m,l)$. For $m \in S_2$, define $f_m : \mathcal{P}(\mathbb{R}) \to ({}^{\omega}\mathbb{R})/\sim_0$ by $Z \mapsto [l \mapsto F(Z,m,l)]_{\sim_0}$, and let $X_m \neq Y_m$ be the canonical pair such that $f_m(X_m) = f_m(Y_m)$. Let l_m be least such that for all $l \geq l_m$, $F(X_m,m,l) = F(Y_m,m,l)$. Let $k = \max_{m\neq n} l_m$. Open all boxes in $\mathcal{P}(\mathbb{R}) \times \{n\} \times (\omega \setminus (k+1))$.

If $F \restriction (X \times \{n\} \times \omega$ and $F \restriction (Y \times \{n\} \times \omega$ have only finitely many discrepancies, then open the box (X,n,k). Guess that the box (Y,n,k) contains $F(X,n,k)$. Otherwise, determine X_n and Y_n as in the previous paragraph, and open (X_n,n,k). Then, guess that the box (Y_n,n,k) contains $F(X_n,n,k)$. This completes the description of strategy σ_n.

To verify that at most one mathematician will fail, consider the number k determined by the nth mathematician; call this k_n. If this mathematician makes an invalid guess, then $k_n < k_m$ for all $m \neq n$. Thus, only one mathematician can fail, completing the argument.

One thing worth emphasizing here is that each player opens the boxes in three waves, as opposed to e.g. making a transfinite sequence of box-opening decisions, each dependent on all the previous ones. This will be useful when more precisely formalizing the box games, since we will want these to be games of finite length.

Generalized box games

From now on, rather than looking at cooperative puzzles involving information unknown to all players, we will consider games between two competing entities, a group called the Allies, and a singular Adversary. The Adversary will select some function F which will not be revealed to the Allies (akin to a collection of boxes), and each Ally will separately query the Adversary for information about this function, without getting to see the queries of the other Allies. A player specifies a query by selecting a set s, and the Adversary must truthfully reveal to that player the function $F \upharpoonright s$. In some versions of the game, the Allies will have a Leader, who will tell the Adversary what the domain of the unknown function must be before the Adversary makes his selection. To fully specify a box game, all that remains is to fix the Ally class (in particular, how many Allies there are), and what the victory condition for the Allies is.

To formalize the observation the Allies can win as long as they have sufficiently many boxes, we will consider the Parley Box Game:

Game 2. *The Ally class is an arbitrary set S containing 0, with Player 0 being the Leader of the Allies. The Adversary begins by selecting a nonempty set Y, after which the Leader selects a nonempty set X. The Adversary then selects a function $F : X \to Y$. Each Ally will make three queries to the Adversary, after which he will select an ordered pair (x, y). If F is unknown on x (i.e., this Ally had not queried any set containing x) and $F(x) = y$, then this is considered a valid guess. The Allies win if at most one fails to make a valid guess.*

Theorem 1. *(Z) The Allies have an explicit winning strategy for Game 2.*

Proof. We will describe a strategy σ as follows: Let Y be the set chosen by the Adversary. The Leader will select $X = S \times \omega \times \kappa$, where $\kappa = H(Y^{S\times\omega})$. [4] We now describe the strategy σ_x Player x uses to make his queries and ultimate guess:

(1) Query the values of F on the set $(S \setminus \{x\}) \times \omega \times \kappa$.

(2) Determine the least $\alpha < \beta < \kappa$ such that there are only finitely many $s \in S$ such that $F \upharpoonright (\{s\} \times \omega \times \{\alpha\}) \neq F \upharpoonright (\{s\} \times \omega \times \{\beta\})$. Let

[4] $H(A)$ is the Hartogs number of A. In Z, this is constructed as the set $\{E \subset \mathcal{P}(A \times A) :$ E is an equivalence class of same-length well-ordered subsets of $A\}$.

$S_1 = \{s \in S \setminus \{x\} : |\{n < \omega : F((s,n,\alpha)) \neq F((s,n,\beta))\}| = \aleph_0\}$ and $S_2 = S \setminus (S_1 \cup \{x\})$. For each $s \in S_1$, let $\beta < \gamma_s < \gamma_s' < \kappa$ be least such that $F \restriction (\{s\} \times \omega \times \{\gamma_s\})$ and $F \restriction (\{s\} \times \omega \times \{\gamma_s'\})$ have only finitely many discrepancies. Let $k < \omega$ be least such that for all $n \geq k$, $s \in S_1$, and $t \in S_2$, that $F((s,n,\gamma_s)) = F((s,n,\gamma_s'))$ and $F((t,n,\alpha)) = F((t,n,\beta))$. Query F on the set $\{x\} \times (\omega \setminus (k+1)) \times \kappa$.

(3) If $F \restriction (\{x\} \times \omega \times \{\alpha\})$ and $F \restriction (\{x\} \times \omega \times \{\beta\})$ have only finitely many discrepancies, then query $\{(x,k,\alpha)\}$. Otherwise, let $\beta < \gamma_x < \gamma_x' < \kappa$ be least such that $F \restriction (\{x\} \times \omega \times \{\gamma_x\})$ and $F \restriction (\{x\} \times \omega \times \{\gamma_x'\})$ have only finitely many discrepancies. Query F on the set $\{(x,k,\gamma_x)\}$.

(4) If in the previous step you queried $\{(x,k,\alpha)\}$, then guess that $F((x,k,\beta)) = F((x,k,\alpha))$. Otherwise, guess that $F((x,k,\gamma_x')) = F((x,k,\gamma_x))$.

In step (2), such α and β exist because there are in fact ordinals[5] $\alpha' < \beta'$ such that $F \restriction (S \times \omega\{\alpha'\}) = F \restriction (S \times \omega\{\beta'\})$. Otherwise, we would have an injection from $H(Y^{S\times\omega})$ to $Y^{S\times\omega}$, contradiction. Same goes for the ordinals γ_s and γ_s'. Thus each step of this strategy is well-defined.

Let k_x be the number k determined by Player x in step (2). If Player x makes an invalid guess, then $k_x < k_y$ for all $y \neq x$. Thus, only one player can fail to make a valid guess. □

It is also interesting to consider two-player versions of the box game, e.g. the following variation of Game 1:

Game 3. *Player 1 selects a function $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$. Player 2 will make three queries to Player 1, after which she will select an ordered pair (x,y). Player 2 wins iff F is unknown on x and $F(x) = y$.*

Theorem 2. *(Z_3P) There exist strategies $\{\sigma_n\}_1^\infty$ for Player 2 such that for any fixed first move $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, at most one σ_i will result in a loss for Player 2.*

Proof. The strategies σ_i are as in the solution to Game 1. □

Thus, Player 1 seems to be able to win Game 3 with a "1 in continuum"[6] chance of failure, simply by selecting F through $\beth_2$ coin flips. However,

[5] We will use the term "ordinals" to describe elements of a specified well-ordered set.

[6] I use this awkward phrasing to informally differentiate between different levels of infinitessimally likely tasks. So guessing one object out of κ many possibilities has a "1 in κ" chance of success.

Player 2 seems to be able to win this game with a less than $\frac{1}{n}$ chance of failure for any fixed $n > 0$ by performing n coin flips to select one of $\{\sigma_i\}_1^{2^n}$ at random. If both players use these nondeterministic strategies against one another, we seem to get a paradox.

Deriving combinatorial theorems from box games

This section is an early attempt at formalizing the severity of paradoxes. I doubt I'll further examine the combinatorial concepts I've defined here, but someone else might find them interesting.

Let's prove some combinatorial theorems using box games and the above strategies. First, we must isolate a combinatorial property that formalizes one of our basic intuitions about probability:

Definition 1. *Let S be a nonempty set and κ a well-ordered cardinal, i.e. an infinite well-ordered set which is of strictly greater cardinality than any of its proper initial segments. Let $X \subset \kappa$ be such that $otp(X) = otp(\kappa \setminus X) = \kappa$. Any $T \in \mathcal{P}(S \times \kappa)$ can be identified with a $(T_X, T_Y) \in (\mathcal{P}(S \times \kappa))^2$ using X and $Y = \kappa \setminus X$ as coordinates. A family $I \subset \mathcal{P}(\mathcal{P}(X \times \kappa))$ is closed under integration if for any X and Y as above and $\mathcal{F} \subset \mathcal{P}(S \times \kappa)$ such that each vertical section $\{T_2 : (T_1, T_2) \in \mathcal{F}\} \in I$, then $\mathcal{F} \in I$.*

One should think of I as consisting of small sets (e.g., I being an ideal), and the intuition is that if $\mathcal{F}$ restricted to any X-coordinate is small, then $\mathcal{F}$ itself is small. We expect that families of sufficiently negligible sets should be closed under integration, e.g. sets representing events as rare as flipping a coin infinitely many times and getting heads each time. We also expect that such events should be closed under finite union, at the very least. However, these two properties are incompatible.

Theorem 3. *(Z) Let S be a nonempty set. There exists a well-ordered cardinal κ such that there is no ideal $I \subset \mathcal{P}(\mathcal{P}(S \times \kappa))$ which*

(1) contains all sets of the form $\{T \subset S \times \kappa : (s, \alpha) \in T \leftrightarrow s \in S'\}$, where $\alpha < \kappa$ and $S' \subset S$;

(2) is closed under integration.

Proof. Let κ be the Hartogs number of $2^{S \times \omega}$. Suppose $I \subset \mathcal{P}(\mathcal{P}(S \times \kappa))$ is an ideal satisfying (1) and (2). Fix $T \subset S \times \kappa$. Let $\alpha < \beta < \kappa$ be minimal such that the functions $f_\alpha, f_\beta : \omega \to \mathcal{P}(S)$ have the same tail, where $f_\gamma(n) = \{s \in S : (s, \omega \cdot \gamma + n) \in T\}$. We define $g : \mathcal{P}(S \times \kappa) \to \omega$ by letting $g(T)$ be minimal such that $f_\alpha(m) = f_\beta(m)$ for all $m \geq n$.

I claim that for all $n < \omega$, the set $G_n := \{T \subset S \times \kappa : g(T) \leq n\} \in I$. We integrate with respect to $X = \{\omega \cdot \alpha + n + m : \alpha < \kappa \wedge m > 0\}$. Fixed $T_1 \subset S \times X$. Let $\alpha < \beta < \kappa$ be minimal such that the functions

$f_\alpha, f_\beta : \omega \to \mathcal{P}(S)$ have the same tail. If $(T_1, T_2) \in G_n$, then $f_\alpha(n) = f_\beta(n)$, so by another application of integration, $\{T_2 : (T_1, T_2) \in G_n\} \in I$. This implies $G_n \in I$.

Let $X \subset \kappa$ be the set of even ordinals, and then Y is the set of odd ordinals. By integration, $\mathcal{F}_1 := \{T \subset S \times \kappa : g(T_X) \leq g(T_Y)\} \in I$, and by integrating with respect to Y, we have $\mathcal{F}_2 := \{T \subset S \times \kappa : g(T_Y) \leq g(T_X)\} \in I$. Then, $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{P}(S \times \kappa) \in I$, contradiction. $\square$

Actually, closure under integration behaves so poorly that we will need a new term to describe this phenomenon:

Definition 2. *Let φ be a closure property and let S and T be nonempty sets. Suppose that if $I \subset \mathcal{P}(\mathcal{P}(S \times T))$ satisfies φ and contains every set of the form $\{U \subset S \times T : (s,t) \in U \leftrightarrow s \in S'\}$, where $t \in T$ and $S' \subset S$, then there exists $\{\mathcal{F}_s\}_{s \in 2^S} \subset I$ such that for every $U \subset S \times T$, there exists $s \in 2^S$ such that $U \in \bigcap_{t \in 2^S \setminus \{s\}} \mathcal{F}_t$. Under these circumstances, we call φ a $\mathcal{P}(S)$-bad closure property.*

Very informally, if φ is a $\mathcal{P}(S)$-bad closure property, that means closing events which have "1 in $2^{|S|}$" probability of occurring under φ results in events which have "1 in $2^{|S|}$" chance of *not* occurring.

Theorem 4. *(Z) Closure under integration is $\mathcal{P}(S)$-bad for every nonempty set S.*

Proof. Let $T = 2^S \times \omega \times \kappa$, where $\kappa = H(2^{S \times 2^S \times \omega})$. Consider Game 2 with set of Allies 2^S, $Y = 2^S$, and $X = 2^S \times \omega \times \kappa$. Let $I \subset \mathcal{P}(\mathcal{P}(S \times T))$ be closed under integration and contain every set of the form $\{U \subset S \times T : (s,t) \in U \leftrightarrow s \in S'\}$, where $t \in T$ and $S' \subset S$. For $t \in 2^S$, let $\mathcal{F}_t$ be the set of $U \subset S \times T$ such that if the Adversary chooses the function $F : (A, n, \alpha) \mapsto \{s \in S : (s, A, n, \alpha) \in U\}$, then σ_t from the proof of Theorem 1 will provide a correct guess. It is clear that for every $U \subset S \times T$, there exists $s \in S$ such that $U \in \bigcap_{t \in S \setminus \{s\}} \mathcal{F}_t$ (or else there would be two Allies who fail to make a valid guess). It suffices to show that each $\mathcal{F}_t \in I$. But this is clear by repeated integrations: no matter what the result of the three queries are, the resulting vertical section of $\mathcal{F}_t$ is in I. We then see the same holds for the first two queries, then the first query, and then we have that $\mathcal{F}_t \in I$. $\square$

Thus, if we go beyond transfinite sequences of coin flips and accept that it is meaningful to select a random subset of any fixed S, then we have that the closure of arbitrarily unlikely events under integration can be arbitrarily likely. This can be demonstrated by the 2-player version of the Game 2, where for some large infinite S, Player 1 chooses $Y = \mathcal{P}(S)$, and Player 2 chooses randomly from a set of $2^{|S|}$ strategies such that only one can fail. If Player 1 chooses $F : X \to Y$ using coin flips, then we have a game in which both players only have a "1 in $2^{|S|}$" chance of failure. In the next section, we will try to 1-up this phenomenon even further and discuss some games with proper classes which demonstrate that in an informal sense, closure under integration is "$\mathcal{P}$(Ord)-bad."

Box games with proper classes

In this section we will discuss one more curiosity, puzzles for which, informally speaking, the chance of success is "absolutely infinitesimal" (i.e., less than ϵ chance of success for any surreal $\epsilon > 0$, with the strategy seemingly irrelevant), but which can in fact be beaten with arbitrarily high probability. We will discuss several versions of the box game which involve proper classes, and naturally, our base theory here will be NBG set theory (without choice) rather than Z.

First we will identify two models of NBG in which extreme versions of the box game are beatable, and then we will combine the ideas from these two games to identify a proper class box game which is provably beatable in NBG by an explicit strategy.

Assuming Global Choice, we can easily generalize the strategy from the original 100-player box game to one with an arbitrary set of Allies and V the class of values. We will go a step further and consider models of NBG (in fact MK) in which there is a definable well-ordering *on the classes*, e.g. $M = \mathcal{P}^L(L_\kappa)$ for some inaccessible κ. This will allow us to consider a box game in which each box contains an arbitrary class:

Game 4. *The Ally class is V. The Adversary begins by selecting a class-valued function F on V. Each Ally will make two queries to the Adversary, after which he will select a double (x, Y), which is valid iff x is an unknown value of F and $F(x) = Y$. The Allies win if at most one fails to make a valid guess.*

Theorem 5. *(NBG) If there is a definable well-ordering $<$ on the classes, then the Allies have an explicit winning strategy for Game 4.*

Proof. Using the definable well-ordering restricted to V and the rank function, we can identify the players with the ordinals. Player α will use the strategy σ_α as follows:

(1) Query the values of F on the class $(\mathrm{Ord} \setminus \{\alpha\}) \times \omega$.

(2) Determine the $<$-least class-valued function $\hat{F}$ which agrees with F on $\{\beta\} \times \omega$ for cofinitely many β. For each $\beta \neq \alpha$, let n_β be least such that $\hat{F}$ agrees with F on $\{\beta\} \times (\omega \setminus n_\beta)$, if such an n exists. Otherwise, let F_β be the $<$-least function which agrees with F cofinitely on $\{\beta\} \times \omega$, and let n_β be least such that F_β agrees with F on $\{\beta\} \times (\omega \setminus n_\beta)$. Let $k = \max_{\beta \neq \alpha} n_\beta$. Query F on $\{\alpha\} \times (\omega \setminus (k+1))$.

(3) If $\hat{F}$ agrees with F cofinitely often on $\{\alpha\} \times \omega$, then guess the double $((\alpha, k, \hat{F}((\alpha, k)))$. Otherwise, let F_α be the $<$-least function which agrees with F cofinitely on $\{\beta\}\times\omega$, and guess the double $((\alpha, k), F_\alpha((\alpha, k)))$.

In step (2), k is well-defined since there are only finitely many β such that $n_\beta \neq 0$. Call this number k_α. Notice that if Player α makes an invalid guess, then $k_\alpha < k_\beta$ for all $\beta \in \mathrm{Ord} \setminus \{\alpha\}$. Clearly only one player can make an invalid guess. $\square$

Now we search for a version of the box game suitable for choiceless models of NBG. We begin with the following notion:

Definition 3. *Let S be a nonempty set. We will call a family $\mathcal{F} \subset \mathcal{P}(S)$ small if there exists $Y \subset S$ such that $\mathcal{F} \cap (\mathcal{F} + Y) = \emptyset$, where $\mathcal{F} + Y = \{X \triangle Y : X \in \mathcal{F}\}$. We will denote the set of small families of subsets of S as $\mathcal{Q}(S)$. Otherwise, there are many subsets of S in $\mathcal{F}$.*

Intuitively, a random subset X of S (produced by flipping a coin at each element of S) is equally likely to be in $\mathcal{F}$ or $\mathcal{F} + Y$ for any $Y \subset S$, since the "event" of X being in $\mathcal{F}$ is equivalent to the event that switching the coin flip at each element of Y results in a set $X' \in \mathcal{F} + Y$. Therefore, if $\mathcal{F} \in \mathcal{Q}(S)$, we expect a random subset of S to be in $\mathcal{F}$ with probability at most $\frac{1}{2}$.

Another notion we will use is the following:

Definition 4. *Fix a class $X \subset$ Ord and an ordinal λ. The family of subsets of λ which appear in X is the set $X_\lambda = \{\{\beta < \lambda : \alpha + \beta \in X\} : \alpha \in \mathit{Ord}\}$. Similarly, given a family $\mathcal{X}$ of subclasses of Ord, the family of subsets of λ which appear in $\mathcal{X}$ is the set $\mathcal{X}_\lambda = \bigcup_{X\in\mathcal{X}} X_\lambda$.*

This allows us to prove our main fact for analyzing box games with proper classes in the case that the Axiom of Global Choice (GC) fails:

Proposition 1. *(NBG) Suppose there is a well-ordered class $\mathcal{X}$ of subclasses of Ord such that many subsets of λ appear in $\mathcal{X}$ for unboundedly many λ, i.e. there is κ such that for all $\lambda > \kappa$, $\mathcal{X}_\lambda \in \mathcal{Q}(\lambda)$. Then GC is true.*

Proof. We prove this proposition by construction. Let $\mathcal{X}$ be as above. It suffices to show that for every ordinal $\delta \in$ Ord, we can canonically well-order $\mathcal{P}(\delta)$ from $\mathcal{X}$. Let $\lambda \geq \delta$ be least such that $\mathcal{X}_\lambda \notin \mathcal{Q}(\lambda)$. There is a natural well-ordering on $\mathcal{X}_\lambda$, and $(X, Y) \mapsto X \triangle Y$ is a surjection from $(\mathcal{X}_\lambda)^2$ to $P(\lambda) \supset \mathcal{P}(\delta)$, providing us a canonical well-ordering of $\mathcal{P}(\delta)$. $\square$

Now we can introduce our second box game with proper classes.

Game 5. *The Ally class is Ord. The Adversary begins by selecting a function $F : V \to V$. Each Ally will make four queries to the Adversary (each query can be a proper class), after which he will select a proper class C of unknown values of F, a nonempty set S, and function $G : C \to \mathcal{Q}(S)$. The triple (C, S, G) is a valid guess iff for all $x \in C$, $F(x) \cap S \in G(x)$. The Allies win if at most one fails to make a valid guess.*

Basically, each Ally will have to make a proper class C of completely independent guesses. If, for example, the Adversary's strategy is to map every x to a random subset of $V_{\mathrm{rk}(x)}$, then for class-many $x \in C$ (particularly all x with $\mathrm{rk}(x) > \mathrm{rk}(S)$), the Ally's guess at x has at most a $\frac{1}{2}$ chance of being correct by the smallness condition (in an informal sense). So it seems each Ally has only an "absolutely infinitesimal" chance of success, no matter the strategy. Yet, we have the following result:

Theorem 6. *(NBG+ $\neg$ GC) The Allies have an explicit winning strategy for Game 5.*

Proof. For $\alpha, \beta \in \mathrm{Ord}$, let $\mathcal{X}^{\alpha,\beta} = \{F((\alpha, \beta, \gamma)) \cap \mathrm{Ord}\}_{\gamma \in \mathrm{Ord}}$. Player α will use the strategy σ_α as follows:

(1) Query the values of F on the class $(\mathrm{Ord} \setminus \{\alpha\}) \times \mathrm{Ord} \times \mathrm{Ord}$.

(2) Determine the least pair (under Gödel's ordering) of limit ordinals (κ, λ) such that there exists a finite set $s \subset \mathrm{Ord}$ such that $\bigcup_{\beta \in \mathrm{Ord} \setminus s, \gamma < \kappa} \mathcal{X}^{\beta,\gamma}_\lambda \in \mathcal{Q}(\lambda)$, and that for all $\beta \in \mathrm{Ord} \setminus s$ and $S \in \bigcup_{n<\omega} \mathcal{X}^{\beta,\kappa+n}_\lambda$, the class $\{\gamma \in \mathrm{Ord} : \exists \delta < \kappa (S \in \mathcal{X}^{\gamma,\delta}_\lambda)\}$ either contains β or is infinite. Query F on the class $\{\alpha\} \times \kappa \times \mathrm{Ord}$.

(3) Using the standard well-ordering of $\mathrm{Ord}^{<\omega}$, determine the first set $s \in \mathrm{Ord}^{<\omega}$ such that $\bigcup_{\beta \in \mathrm{Ord} \setminus s, \gamma < \kappa} \mathcal{X}^{\beta,\gamma}_\lambda \in \mathcal{Q}(\lambda)$. Let $\mathcal{F} = \bigcup_{\beta \in \mathrm{Ord} \setminus s, \gamma < \kappa} \mathcal{X}^{\beta,\gamma}_\lambda$. Let $S_1 = \{\beta \in \mathrm{Ord} \setminus \{\alpha\} : |\{n < \omega : \mathcal{X}^{\beta,\kappa+n}_\lambda \not\subset \mathcal{F}\}| = \aleph_0\}$, and let $S_2 = \mathrm{Ord} \setminus (S_1 \cup \{\alpha\})$. For $\beta \in S_1$, determine the least pair of limit ordinals $(\kappa_\beta, \lambda_\beta)$ such that $\kappa_\beta > \kappa$ and that for infinitely many $n < \omega$, $\bigcup_{m \in \omega \setminus n} \mathcal{X}^{\beta,\kappa_\beta+m}_{\lambda_\beta} \subset \bigcup_{m \in \omega \setminus n, \kappa \le \delta < \kappa_\beta} \mathcal{X}^{\beta,\delta+m}_{\lambda_\beta} \in \mathcal{Q}(\lambda_\beta)$. Let $n_\beta < \omega$ be least such that $\bigcup_{m \in \omega \setminus n_\beta, \kappa \le \delta < \kappa_\beta} \mathcal{X}^{\beta,\delta+m}_{\lambda_\beta} \in \mathcal{Q}(\lambda_\beta)$.

Let $k < \omega$ be least such that for all $n \geq k$, $\beta \in S_1$, and $\gamma \in S_2$, that $\mathcal{X}^{\beta,\kappa_\beta+k}_{\lambda_\beta} \subset \bigcup_{m\in\omega\setminus n_\beta,\kappa\leq\delta<\kappa_\beta} \mathcal{X}^{\beta,\delta+m}_{\lambda_\beta}$ and $\mathcal{X}^{\gamma,\kappa+n}_{\lambda} \subset \mathcal{F}$. Query F on the class $\{\alpha\} \times \{\delta + k + 1 : \delta \geq \kappa\} \times \text{Ord}$.

(4) If $\{n < \omega : \mathcal{X}^{\alpha,\kappa+n}_{\lambda} \not\subset \mathcal{F}\}$ is finite, query the empty set. Otherwise, determine the pair $(\kappa_\alpha, \lambda_\alpha)$, analagously to the pairs $(\kappa_\beta, \lambda_\beta)$ in the previous step. Query F on $\{\alpha\} \times (\kappa_\alpha \setminus \kappa) \times \text{Ord}$.

(5) If you made the empty query in the previous step, then guess the triple $(\{\alpha\} \times \{\kappa + k\} \times \text{Ord}, \lambda, G)$, where G is the constant function $x \mapsto \mathcal{F}$. Otherwise, determine n_α analogously to n_β in the third step, and guess the triple $(\{\alpha\} \times \{\kappa_\alpha + k\} \times \text{Ord}, \lambda_\alpha, G_\alpha)$, where G_α is the constant function $x \mapsto \bigcup_{m\in\omega\setminus n_\alpha,\kappa\leq\delta<\kappa_\alpha} \mathcal{X}^{\alpha,\delta+m}_{\lambda_\alpha}$.

Notice that the pair (κ, λ) determined after the first query does not depend on α. Let's check that such a pair exists. Let λ' be the least limit ordinal such that $\bigcup_{\alpha,\beta\in\text{Ord}} \mathcal{X}^{\alpha,\beta}_{\lambda'} \in \mathcal{Q}(\lambda')$, which exists by Proposition 1 and $\neg$ GC. For each $S \subset \lambda'$, there are only countably many limit ordinals δ such that there is $\beta \in \text{Ord}$ such that $S \in \bigcup_{n<\omega} \mathcal{X}^{\beta,\delta+n}_{\lambda'}$ but the class $\{\gamma \in \text{Ord} : \exists\delta' < \delta(S \in \mathcal{X}^{\gamma,\delta'}_{\lambda'})\}$ is finite and does not contain β. Thus, there is a least limit ordinal $\delta = \kappa'$ for which no such S exists. Then (κ', λ') is a pair with the prescribed properties, so there is a least such pair.

Also it is easily checked that every piece of information Player α is required to find can be determined from the information he has acquired in the previous queries. Finally, we check that $\sigma := \{\sigma_\alpha : \alpha \in \text{Ord}\}$ is a winning strategy. Let k_α be the number k determined by Player α in the third step. Notice that if Player α makes an invalid guess, then $k_\alpha < k_\beta$ for all $\beta \in \text{Ord} \setminus \{\alpha\}$. Clearly only one player can make an invalid guess. □

Corollary 1. *There is a model of NBG in which the Allies have an explicit winning strategy for Game 5 played with Ally Class V.*

Proof. Notice that if Game 5 is played with Ally class $S \times \text{Ord}$ for an arbitrary nonempty set S, the game is still beatable. The winning strategy is a generalization of the strategy for $S = \{1\}$, beginning with the selection of the least pair (κ, λ) such that there is a finite set $s \subset S \times \text{Ord}$ which if removed, for any $x \in S$, the class $\{x\} \times (\text{Ord})^3$ satisfy the requirements with respect to κ and λ as in the second step of the above strategy. The remaining details follow naturally.

Now consider a model of NBG in which $V = L(\mathbb{R})$ and there is no well-ordering of $\mathbb{R}$. This model satisfies $\neg$ GC, so we have a winning strategy for Ally Class $\mathcal{P}(\mathbb{R}) \times \text{Ord}$. Since there is a definable injection from $V \to \mathcal{P}(\mathbb{R}) \times \text{Ord}$, this gives us an explicit winning strategy for Ally Class V. $\square$

Finally, we define a proper class box game which combines the win conditions of Games 4 and 5, so that NBG proves there is an explicit winning strategy.

Game 6. *The Ally class is an arbitrary ordinal η. The Adversary begins by selecting a function $F : V \to V$. Each Ally will make three queries to the Adversary (each query can be a proper class), after which he will either guess*

(a) a double (x, y), which is valid iff x is an unknown value of F and $F(x) = y$, or

(b) a triple (C, S, G), which is valid iff C is a proper class of unknown values of F, S is nonempty, $G : C \to \mathcal{Q}(S)$, and for all $x \in C$, $F(x) \cap S \in G(x)$.

The Allies win if at most one fails to make a valid guess.

Here each Ally has two ways to make a guess, but either way, it still seems each one has an "absolutely infinitesimal" chance of success.

Theorem 7. *(NBG)*[7] *The Allies have an explicit winning strategy for Game 6.*

Proof. We will describe the strategy σ_α of Player α. On the first turn, query F on $(\eta \setminus \{\alpha\}) \times \text{Ord} \times \text{Ord} \cup (\eta \setminus \{\alpha\}) \times \omega$.

Now we will consider two cases from a global perspective, while making it clear that none of the Allies are relying on information beyond what they have queried. Let $\mathcal{X}^{\alpha,\beta} = \{F((\alpha, \beta, \gamma)) \cap \text{Ord}\}_{\gamma \in \text{Ord}}$. For a fixed $\beta < \eta$, let (*) be the property that for unboundedly many $\lambda \in \text{Ord}$, $\bigcup_{\gamma \in \text{Ord}} \mathcal{X}^{\beta,\gamma}_\lambda \notin \mathcal{Q}(\lambda)$.

Case 1: The are infinitely many $\beta < \eta$ which satisfy (*).

Every player is aware they are in Case 1. On the second query, Player α queries F on Ord^3. By the proof of Proposition 1, the players now have a

[7] This result is in fact formalizable in ZFP, set theory with a unary predicate P and the replacement scheme extended to include formulae which use the predicate. The predicate represents the function selected by the Adversary.

canonical well-ordering $<$ of V. Determine the $<$-first $G : \eta \times \omega \to V$ which agrees with F except on finitely many sections $\{\beta\} \times \omega$.

Let $S_1 = \{\beta \in \eta \setminus \{\alpha\} : |\{F((\beta, n)) \neq G((\beta, n))\}| = \aleph_0\}$ and let $S_2 = \eta \setminus (S_1 \cup \{\alpha\})$. Let $k < \omega$ be minimal such that for all $m \geq n$, $\beta \in S_1$, and $\gamma \in S_2$, that $F((\gamma, m)) = G((\gamma, m))$, and that the $<$-least function $G_\beta : \{\beta\} \times \omega \to V$ which agrees with F cofinitely often agrees with F on m. Query F on $\{\alpha\} \times (\omega \setminus (k + 1))$.

If F agrees with G cofinitely often on $\{\alpha\} \times \omega$, then guess that $F((\alpha, k)) = G((\alpha, k))$. Otherwise, let $G_\alpha : \{\alpha\} \times \omega \to V$ be the $<$-first function which agrees cofinitely often with F. Guess that $F((\alpha, k)) = G_\alpha((\alpha, k))$.

Case 2: There are only finitely many $\beta < \eta$ which satisfy (*).

We will split this into three subcases. For a fixed $\beta < \eta$, let (**) be the property that for every $n < \omega$, there are unboundedly many $\lambda \in \mathrm{Ord}$ such that $\bigcup_{m > n, \gamma \in \mathrm{Ord}} \mathcal{X}_\lambda^{\beta, \gamma + m} \notin \mathcal{Q}(\lambda)$.

Case 2a: There are at least two $\beta < \eta$ which satisfy (**).

In this case, every player sees there is at least one β which satisfies (**). On the second query, Player α queries F on Ord^3. Now everyone knows they are in Case 2a. By the proof of Proposition 1, the players now have a canonical well-ordering of V. Everyone proceeds as in Case 1.

Case 2b: There are no $\beta < \eta$ which satisfy (**).

In this case, of course no player sees any β which satisfies (**). We proceed similarly to the strategy for Game 5:

(1) Let (κ, λ) be the first pair of limit ordinals such that for cofinitely many $\beta < \eta$, $\bigcup_{n < \omega} \mathcal{X}_\lambda^{\beta, \kappa + n} \subset \bigcup_{\gamma < \kappa} \mathcal{X}_\lambda^{\beta, \gamma} \in \mathcal{Q}(\lambda)$. Let $S_1 = \{\beta \in \eta \setminus \{\alpha\} : \bigcup_{\gamma < \kappa} \mathcal{X}_\lambda^{\beta, \gamma} \notin \mathcal{Q}(\lambda)\} \cup \{\beta \in \eta \setminus \{\alpha\} : |\{n < \omega : \mathcal{X}_\lambda^{\beta, \kappa + n} \not\subset \bigcup_{\gamma < \kappa} \mathcal{X}_\lambda^{\beta, \gamma}\}| = \aleph_0\}$, and let $S_1 = \eta \setminus (S_2 \cup \{\alpha\})$. For $\beta \in S_1$, determine the least pair of limit ordinals $(\kappa_\beta, \lambda_\beta)$ such that $\kappa_\beta > \kappa$ and that for infinitely many $n < \omega$, $\bigcup_{m \in \omega \setminus n} \mathcal{X}_{\lambda_\beta}^{\beta, \kappa_\beta + m} \subset \bigcup_{m \in \omega \setminus n, \kappa \leq \delta < \kappa_\beta} \mathcal{X}_{\lambda_\beta}^{\beta, \delta + m} \in \mathcal{Q}(\lambda_\beta)$. Let $n_\beta < \omega$ be least such that $\bigcup_{m \in \omega \setminus n_\beta, \kappa \leq \delta < \kappa_\beta} \mathcal{X}_{\lambda_\beta}^{\beta, \delta + m} \in \mathcal{Q}(\lambda_\beta)$.

   Let $k < \omega$ be least such that for all $n \geq k$, $\beta \in S_1$, and $\gamma \in S_2$, that $\mathcal{X}_{\lambda_\beta}^{\beta, \kappa_\beta + k} \subset \bigcup_{m \in \omega \setminus n_\beta, \kappa \leq \delta < \kappa_\beta} \mathcal{X}_{\lambda_\beta}^{\beta, \delta + m}$ and $\mathcal{X}_\lambda^{\gamma, \kappa + n} \subset \bigcup_{\gamma < \kappa} \mathcal{X}_\lambda^{\beta, \gamma}$. On the second query, Player α queries F on $\{\alpha\} \times (\kappa \cup \{\delta + k + 1 : \delta \geq \kappa\}) \times \mathrm{Ord}$.

(2) Now all the players know they are in Case 2b. If $\{n < \omega : \mathcal{X}_\lambda^{\alpha, \kappa + n} \not\subset \bigcup_{\gamma < \kappa} \mathcal{X}_\lambda^{\alpha, \gamma}\}$ is finite, Player α queries the empty set. Otherwise, determine the pair $(\kappa_\alpha, \lambda_\alpha)$, analagously to the pairs $(\kappa_\beta, \lambda_\beta)$ in the previous step. Query F on $\{\alpha\} \times (\kappa_\alpha \setminus \kappa) \times \mathrm{Ord}$.

(3) If Player α made the empty query in the previous step, then guess the triple $(\{\alpha\} \times \{\kappa + k\} \times \mathrm{Ord}, \lambda, G)$, where G is the constant function $x \mapsto \bigcup_{\gamma<\kappa} \mathcal{X}_{\lambda}^{\alpha,\gamma}$. Otherwise, determine n_α analogously to n_β in step 1, and guess the triple $(\{\alpha\} \times \{\kappa_\alpha + k\} \times \mathrm{Ord}, \lambda_\alpha, G_\alpha)$, where G_α is the constant function $x \mapsto \bigcup_{m\in\omega\setminus n_\alpha, \kappa\leq\delta<\kappa_\alpha} \mathcal{X}_{\lambda_\alpha}^{\alpha,\delta+m}$.

Case 2c: There is a unique $\beta < \eta$ which satisfies (**).

First let's suppose $\alpha = \beta$, so Player α does not see any ordinal which satisfies (**). He determines κ, λ, and k and does his second query as in Case 2b. Now he has enough information to determine that α satisfies (**), and thus he is in Case 2c. Player α constructs a well-ordering of V from $\bigcup_{\gamma\in\mathrm{Ord}} \mathcal{X}_{\lambda}^{\alpha,\gamma+k+1}$.

Now suppose $\alpha \neq \beta$. Player α sees that β satisfies (**), so on his second query, he queries F on Ord^3 as in Case 2a. He sees that he is in Case 2c. Knowing F on Ord^3, Player α can determine which well-ordering of V will be constructed by Player β. Thus, after the second query, all players obtain the same well-ordering of V, and can proceed as in Case 1.

Verification: Having described all cases, we verify that this is a winning strategy. Let k_α be the number k determined by Player α. Notice that if Player α makes an invalid guess, then $k_\alpha < k_\beta$ for all $\beta \in \eta \setminus \{\alpha\}$. Clearly only one player can make an invalid guess. □

Other box games

Clearly there are many possible box games, though I believe I've identified the most interesting winnable variants, so let's discuss some negative results, with briefly sketched proofs.

We'll constrain our attention to set-sized variants of Game 1. Namely, let $G(A, X, Y)$ be the box game with Ally set A, box set X, and set of values Y, e.g. Game 1 is $G(\omega, \mathcal{P}(\mathbb{R}), \mathbb{R})$. We won't limit the number of queries the Allies can make (they can even make a transfinite sequence of queries).

First we'll show that the Ally set must inject into the box set for the game to be beatable.

Proposition 2. *(Basic set theory) If $|X|, |Y| \geq 2$ and $|A| \not\leq |X|$, then there is no winning strategy for $G(A, X, Y)$.*

Proof. Suppose A, X, Y, and strategies $\{\sigma_\alpha\}_{\alpha \in A}$ comprise a counterexample. We can assume wlog that $Y = 2$. Consider the Allies' actions when every box has 0 in it. We make the following deductions:

(1) No 3 Allies guess from the same box (or else we could change the value in that box to 1, and at least 2 Allies would have been wrong).

(2) If 2 Allies guess from the same box, they make different guesses.

(3) At most 1 box gets guesses from 2 Allies.

(4) Exactly 1 box gets guesses from 2 Allies, and all the others get guesses from exactly 1 Ally (using $|A| \not\leq |X|$).

(5) Exactly 1 Ally guesses wrong, and it's one of the 2 Allies who shares a box with another.

If we change the value in one of the non-shared boxes to 1, then the Ally who guesses that box gets it wrong. By the above argument, some other Ally would also have to guess from that box. But that's impossible, since no other Ally would act differently in performing their strategy until after opening that box.

□

It's obvious from probability theory that there is no winning strategy for any nontrivial box game with finitely many boxes. In ZFC, these are the only constraints on which box games can be beaten, since $G(\kappa, \kappa, \lambda)$ is always beatable. Now we demonstrate that choice is necessary to prove that the original box game with countably many boxes is beatable.

Proposition 3. *(ZF + LM + DC) The games* $G(3, \omega, 2)$ *and* $G(2, \omega, 3)$ *don't have winning strategies for the Allies.*

Proof. We'll just consider $G(3, \omega, 2)$. Assume the values in the boxes are chosen independently, uniformly at random. We compute the probability of success for a single strategy σ. There is some countable α such that σ will almost certainly terminate in less than α stages. The space of $F : \omega \to 2$ can be factored into the space of possible outputs of an α-sequence of queries, cross the possible values in the remaining boxes. No matter what outputs are shown to the player, they have a $\frac{1}{2}$ probability of making a valid guess (or 0 if they guess on an already opened box), so the probability σ succeeds is at most $\frac{1}{2}$. Therefore, the expected the numbers of failures among 3 strategies is at least $\frac{3}{2}$, so there is some F for which there are at least 2 failures. □

We should be able to generalize this to show that ZF doesn't prove there is any non-trivial winnable box game with $\alpha < H(2^{\aleph_0})$ boxes using translation-invariant total measures on $\mathcal{P}(2^\alpha)$, though I haven't worked this out carefully.

The most interesting open question is whether ZF proves there is a non-trivial winnable box game with a continuum of boxes, e.g. the game $G(3, 2^{\aleph_0}, 2)$. Almost certainly it doesn't. I don't even know if this game can be winnable in a model where the countable box games aren't. Resolving this question should provide great insight into whether a continuum of coin flips can be a coherent, non-paradoxical concept.

Conclusions

My conclusions here are completely out of line with my current thoughts on probability, but I'll leave them here for posterity. My current position is roughly that randomness limits the size of infinity, and in particular, we can't believe in both a Platonistic theory of propensities and sets as large as $\mathcal{P}(\mathbb{R})$.

In this note, we have derived multiple probabilistic paradoxes from weak hypotheses, including many results not requiring any choice principle (including one result which uses the *negation* of a choice principle). All of these results are instead based on the principle that families of negligible events should be closed under integration. We argue that this is a false principle, and that many of the paradoxes which are blamed on choice are in fact a result of the failure of this principle. Indeed, the only paradox discussed in this note that doesn't seem to stem from the poor closure properties of integration is the Banach-Tarski paradox, which is more concerned with the additivity of probability. In a future paper, I will argue that the Banach-Tarski paradox and failures of integration are actually instances of a more fundamental failure of probabilistic intuition, that being that probabilities are linearly ordered. For now, it suffices to accept that AC is not the culprit.

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