Galvin's Question on non-σ-well ordered Linear Orders

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Introduction

Definition

A linear order L is said to be σ-well ordered if it is a countable union of well ordered suborders.

Question (Galvin)

Does every non σ-well ordered linear order contain a copy of one of the following?

- a real type
- an Aronszajn type
- ω_1^*

Baumgartner answered this question negatively.

Introduction

Theorem (Baumgartner, 1973)

Assume $\langle C_\alpha : \alpha \in S \rangle$ is a ladder system and $S \subset \omega_1$ is stationary. Let $L = \{C_\alpha : \alpha \in S\}$ ordered with $<_{\text{lex}}$. Then:

- L is not σ-well ordered,
- if $S' \subset S$, $S \setminus S'$ is stationary then L does not embed into $L \restriction S'$.

Question (Galvin)

Assume $\mathcal{C}$ is the class of all linear orders L which are not σ-well ordered and every uncountable suborder of L contains a copy of ω_1. Does $\mathcal{C}$ have minimal elements?

The answer to Galvin's question is independent

Theorem (Ishiu-Moore)

Assume PFA^+. Then every minimal non σ-scattered linear order is either a real type or Aronszajn type.

Theorem

It is consistent that there is a minimal non σ-well ordered linear order which does not contain copies of ω_1^*, real types and Aronszajn types.

Background

Fact

Assume T is a lexicographically ordered ω_1-tree such that $(T, <_{lex})$ has a copy of ω_1^*. Then there is a branch b and a sequence of branches $\langle b_\xi : \xi \in \omega_1 \rangle$ such that:

- *for all $\xi \in \omega_1$, $b <_{lex} b_\xi$*
- $\sup\{\Delta(b, b_\xi) : \xi \in \omega_1\} = \omega_1$.

Definition

- Assume L is a linear order. We use $\hat{L}$ in order to refer to the *completion* of L. In other words, we add all the Dedekind cuts to L in order to obtain $\hat{L}$.
- For any set Z and $x \in L$ we say Z *captures* x if there is $z \in Z \cap \hat{L}$ such that $Z \cap L$ has no element which is strictly in between z and x.

Ω and Γ

Definition

The invariant $\Omega(L)$ is defined to be the set of all countable $Z \subset \hat{L}$ such that Z captures all elements of L. We let $\Gamma(L) = [\hat{L}]^\omega \setminus \Omega(L)$.

Assume T is a lexicographically ordered ω_1-tree such that for every $t \in T$, there is a cofinal branch $b \subset T$ with $t \in b$. By $\Omega(T), \Gamma(T)$ we mean $\Omega(\mathcal{B}(T)), \Gamma(\mathcal{B}(T))$, where $\mathcal{B}(T)$ is considered with the lexicographic order.

Ω and Γ

Assume T is a lexicographically ordered ever branching ω_1-tree such that for every $t \in T$, there is a cofinal branch $b \subset T$ with $t \in b$. Let θ be a regular cardinal such that $\mathcal{P}(T) \in H_\theta$, $M \prec H_\theta$ be countable such that $T \in M$ and $b \in \mathcal{B}(T)$. Then M captures b iff there is $c \in \mathcal{B}(T) \cap M$ such that $\Delta(b, c) \geq M \cap \omega_1$.

Theorem (Ishiu-Moore)

L is σ-scattered iff $\Gamma(L)$ is not stationary in $[\hat{L}]^\omega$.

Another form of Galvin's Question

Question

Does there exist a lexicographically ordered ω_1-tree T such that the following holds?

1. T has no Aronszajn subtree.
2. For any $b \in \mathcal{B}(T)$ there is $\alpha \in \omega_1$ such that if $c \in \mathcal{B}(T)$ and $\Delta(b, c) > \alpha$ then $c <_{\mathrm{lex}} b$.
3. If $L \subset \mathcal{B}(T)$ is nowhere dense then $\Gamma(L)$ is non-stationary.
4. If L is somewhere dense then $\mathcal{B}(T)$ embeds into L.
5. $\Gamma(T)$ is stationary.

Definition of Q

Fix a set Λ of size $\aleph_1$. The forcing Q is the poset consisting of all conditions (T_q, b_q, d_q) such that the following hold.

1. $T_q \subset \Lambda$ is a lexicographically[1] ordered countable tree of height $\alpha_q + 1$ with the property that for all $t \in T_q$ there is $s \in (T_q)_{\alpha_q}$ such that $t \leq_{T_q} s$.
2. b_q is a bijective map from a countable subset of ω_1 onto $(T_q)_{\alpha_q}$.
3. The map $d_q : \mathrm{dom}(b_q) \longrightarrow \omega_1$ has the property that if $b_q(\xi) = t$, $b_q(\eta) = s$ and $t <_{\mathrm{lex}} s$ then $\Delta(t, s) < d_q(\xi)$.

[1] Note that the lexicoraphic order here is independent of any structure on Λ if it exists. In other words, this order which we refer to as $<_{\mathrm{lex}}$ is determined by the condition q.

Order on Q

We let $q \leq p$ if the following hold.

1. $T_p \subset T_q$ and $(T_p)_{\alpha_p} = (T_q)_{\alpha_p}$.
2. For all s, t in T_p, $s <_{\mathrm{lex}} t$ in T_p if and only if $s <_{\mathrm{lex}} t$ in T_q.
3. For all s, t in T_p, $s \leq_{T_p} t$ if and only if $s \leq_{T_q} t$.
4. $\mathrm{dom}(b_p) \subset \mathrm{dom}(b_q)$.
5. For all $\xi \in \mathrm{dom}(b_p)$, $b_p(\xi) \leq_T b_q(\xi)$.
6. $d_p \subset d_q$.

Different kinds of lower bounds

Lemma 1

Assume $\langle q_n : n \in \omega \rangle$ is a decreasing sequence of conditions in Q, $m \leq \omega$ and for each $i \in m$ let $c_i \subset \bigcup_{n \in \omega} T_{q_n}$ be a cofinal branch.

Then there is a lower bound q for the sequence $\langle q_n : n \in \omega \rangle$ in which every c_i has a maximum with respect to the tree order in T_q. Moreover, for every $t \in (T_q)_{\alpha_q}$ either there is $i \in m$ such that t is above all elements of c_i or there is $\xi \in D = \bigcup_{n \in \omega} \mathrm{dom}(b_{q_n})$ such that t is above all elements of $\{b_{q_n}(\xi) : n \in \omega \wedge \xi \in \mathrm{dom}(b_{q_n})\}$. In particular, Q is σ-closed.

Fact

- $\Gamma(T)$ *and* $\Omega(T)$ *are stationary.*
- $\mathcal{B}(T)$ *has copies of Baumgartner types, which we have to kill.*
- *Every uncountable downward closed subset of* T *contains some* b_ξ*.*

Definition

For every $\xi \in \omega_1$, $d(\xi) = \sup\{\Delta(b_\xi, b_\eta) : b_\xi <_{lex} b_\eta\}$, *and if* $b = b_\xi$ *we sometimes use* $d(b)$ *instead of* $d(\xi)$.

Lemma

The function d *is a countable to one function, i.e. for all* $\alpha \in \omega_1$ *there are countably many* $\xi \in \omega_1$ *with* $d(\xi) = \alpha$.

Lemma

For every $t_0 \in T$ *and* $\beta > ht(t)$, *there is an* $\alpha > \beta$ *such that* $(T_\alpha \cap T_{t_0}, <_{lex})$ *contains a copy of the rationals.*

How to make nowhere dense sets σ-well ordered

Definition

Assume $L \subset B$ is nowhere dense. Define S_L to be the poset consisting of all increasing continuous countable sequences $\langle \alpha_i : i \in \beta + 1 \rangle$ such that $\beta \in \omega_1$ and for all i and $t \in T_{\alpha_i} \cap (\bigcup L)$ there is $\xi < \alpha_i$ with $t \in b_\xi$. If $p, q \in S_L$, q is an extension of p if p is an initial segment of q.

Definition

Assume X is uncountable and $S \subset [X]^\omega$ is stationary. A poset P is said to be S-complete if every descending (M, P)-generic sequence $\langle p_n : n \in \omega \rangle$ has a lower bound, for all M with $M \cap X \in S$ and M suitable for X, P.

Definition

Assume $U = T_x$ *for some* $x \in T$ *and* $L \subset \mathcal{B}(U)$ *is dense in* $\mathcal{B}(U)$. *Define* E_L *to be the poset consisting of all conditions* $q = (f_q, \phi_q)$ *such that:*

1. $f_q : T \restriction A_q \longrightarrow U \restriction A_q$ *is a* $<_{lex}$*-preserving tree embedding where* A_q *is a countable and closed subset of* ω_1 *with* $\max(A_q) = \alpha_q$,
2. ϕ_q *is a countable partial injection from* ω_1 *into* $\{\xi \in \omega_1 : b_\xi \in L\}$ *such that the map* $b_\xi \mapsto b_{\phi_q(\xi)}$ *is* $<_{lex}$*-preserving,*
3. *for all* $t \in T_{\alpha_q}$ *there are at most finitely many* $\xi \in \mathrm{dom}(\phi_q) \cup \mathrm{range}(\phi_q)$ *with* $t \in b_\xi$,
4. f_q, ϕ_q *are consistent, i.e. for all* $\xi \in \mathrm{dom}(\phi_q)$, $f_q(b_\xi(\alpha_q)) \in b_{\phi_q(\xi)}$,
5. *for all* $\xi \in \mathrm{dom}(\phi_p)$, $d(\xi) \le d(\phi_p(\xi))$

We let $q \le p$ *if* A_p *is an initial segment of* A_q, $f_p \subset f_q$, *and* $\phi_p \subset \phi_q$.

Lemma 2

For all $\beta \in \omega_1$ the set of all conditions $q \in E_L$ with $\alpha_q > \beta$ is dense in E_L.

Proof

Fix $p \in E_L$ and let $D_p = \mathrm{dom}(\phi_p)$ and $R_p = \mathrm{range}(\phi_p)$. We sometimes abuse the notation and use D_p, R_p in order to refer to the corresponding set of branches, $\{b_\xi : \xi \in D_p\}$ and $\{b_\xi : \xi \in R_p\}$. We consider the following partition of $U = T_{\alpha_p} \cap \mathrm{range}(f_p)$. Let U_0 be the set of all $u \in U$ such that if $u \in b \in R_p$ then there is a $c \in B$ with $u \in c$ and $b <_{\mathrm{lex}} c$.

If $u \in U_0$ then there is $\alpha_u \in \omega_1$ and $X_u \subset T_{\alpha_u} \cap T_u$ such that:

a. $\alpha_u > \max(\{\Delta(b, c) : b, c \text{ are in } A\} \cup \{\beta\})$, where A is the set of all $b \in R_p$ such that $u \in b$,

b. $(X_u, <_{\text{lex}})$ is isomorphic to the rationals, and

c. $\{b(\alpha_u) : b \in A\} \subset X_u$.

For all $u \in U_1$ there is $\alpha_u \in \omega_1$ and $X_u \subset T_{\alpha_u} \cap T_u$ such that:

d. $\alpha_u > \max(\{\Delta(b, c) : b, c \text{ are in } A\} \cup \{\beta\})$, where A is the set of all $b \in R_p$ such that $u \in b$,

e. $(X_u \setminus \{b_m(\alpha)\}, <_{\text{lex}})$ is isomorphic to the rationals, where b_m is the maximum of A with respect to $<_{\text{lex}}$,

f. $\{b(\alpha_u) : b \in A\} \subset X_u$, and

g. $\max(X_u, <_{\text{lex}}) = b_m(\alpha)$.

Lemma 3

For all $\xi \in \omega_1$, the set of all $q \in E_L$ with $\xi \in \mathrm{dom}(\phi_q)$ is dense in E_L.

Lemma

The forcing E_L is $\Omega(T)$-complete.

Since E_L, S_L are $\Omega(T)$-complete forcings, the countable support iteration consisting of the posets E_L, S_L do not add new branches to T.

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It is easy to see that in $\mathbf{V}^P$, $\mathcal{B}(T)$ satisfies the first four conditions. We need to work for the last condition.

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Assume M is a suitable model for P in $\mathbf{V}$ with $M \cap \omega_1 = \delta$ and $\langle p_n : n \in \omega \rangle$ is a descending (M, P)-generic sequence.

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Note that if $G \subset P$ is a generic filter over $\mathbf{V}$ which contains $\langle p_n : n \in \omega \rangle$, then in $\mathbf{V}[G]$ we have $T_{<\delta} = R$.

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For each $\xi \in \delta$, let e_ξ be the downward closure of $\{b_{q_n}(\xi) : n \in \omega\}$ in R.

We continue the proof on the whiteboard.