Ramsey quantifiers

Stephen G. Simpson

sgslogic@gmail.com

sgslogic.net

Vanderbilt University

Jim Schmerl Celebration

April 22, 2025

Angus Macintyre, *Ramsey quantifiers in arithmetic*, Karpacz 1979, Springer Lecture Notes in Mathematics, 834, 186–210, 1980

James H. Schmerl and Stephen G. Simpson, *On the role of Ramsey quantifiers in first order arithmetic*, Journal of Symbolic Logic, 47, 2, 423–435, 1982.

Let PA be the usual axiom system for first-order Peano arithmetic. The standard model of PA is $\mathbb{N}, +_{\mathbb{N}}, \times_{\mathbb{N}}$. We expand the language of PA by introducing, for each $k \geq 1$, a *Ramsey quantifier* Q^k which binds k number variables.

In the standard model $\mathbb{N}$, the semantics for Q^k is: $\mathsf{Q}^k x_1 \cdots x_k \, \Phi(x_1, \ldots, x_k)$ holds if and only if there exists an infinite set $X \subseteq \mathbb{N}$ such that $\Phi(a_1, \ldots, a_k)$ holds for all $a_1, \ldots, a_k \in X$ such that $a_1 < \cdots < a_k$.

Building on Magidor/Malitz 1977, Macintyre 1979 introduced an axiomatic theory PA(Q) consisting of PA plus some axioms for Ramsey quantifiers, and he proved a completeness theorem for PA(Q). However, his proof used a set-theoretical hypothesis which is independent of ZFC, namely Jensen's diamond.

Schmerl 1980 gave a better set of axioms and a better version of the theorem, and he eliminated the diamond hypothesis. These results were published in §2 of Schmerl/Simpson 1982.

Schmerl's axioms for PA(Q) with $k = 2$:

1. The axioms of PA, plus induction for all PA(Q)-formulas.

2. If $\mathrm{Q}xy\,\Phi(x, y)$ and $\forall x\,\forall y\,(\Phi(x, y) \Rightarrow \Psi(x, y))$, then $\mathrm{Q}xy\,\Psi(x, y)$.

3. If $\mathrm{Q}xy\,\Phi(x, y)$, then $\exists z\,\mathrm{Q}xy\,(z < x \wedge z < y \wedge \Phi(z, x) \wedge \Phi(z, y) \wedge \Phi(x, y))$.

4. If $\forall x\,\forall y\,((x < y \wedge \Theta(x) \wedge \Theta(y)) \Rightarrow \Phi(x, y))$ and $\forall z\,\exists x\,(z < x \wedge \Theta(x))$, then $\mathrm{Q}xy\,\Phi(x, y)$.

Let $S \supseteq$ PA(Q) be a set of sentences in the language of PA(Q). A *strong model of S* is a model M of PA which satisfies S if we interpret $M \models \mathrm{Q}xy\,\Phi(x, y)$ as ($\exists$ <u>unbounded</u> $X \subseteq M)\,(\forall x, y \in X)\,(x < y \Rightarrow \Phi(x, y))$.

Completeness/Compactness Theorem (Schmerl): S is consistent if and only if, for all uncountable regular cardinals κ, S has a κ-like strong model.

In his 1979 paper, Macintyre discusses a famous theorem of Paris and Harrington. Namely, a certain slightly modified version of the finite Ramsey Theorem is not provable in PA, a.k.a. first-order arithmetic. The Paris/Harrington theorem had important foundational implications, because it was the first transparent example of a finite combinatorial statement which is not provable in PA. And a little later, other authors produced similar finite combinatorial statements which are not provable in some of the best-known subsystems of second-order arithmetic which are much stronger than PA.

This situation was the motivation for Macintyre's study of PA(Q), which is also much stronger than PA. It had become desirable to compare PA(Q) with subsystems of second-order arithmetic, from the perspective of finite combinatorial theorems.

In §3 of Schmerl/Simpson 1982, we solved this problem by presenting the following theorem:
A sentence in the language of PA is provable in PA(Q) if and only if it is provable in Π^1_1-CA_0.

This system Π^1_1-CA_0 is now well-known as one of the "Big Five" subsystems of second-order arithmetic, studied in Chapters 2 through 6 of my 1999 book. These five systems are recognized as being important because they play a key role in reverse mathematics.

Other relevant references:

Menachem Magidor and Jerome I. Malitz, *Compact extensions of L(Q)*, Annals of Mathematical Logic, 16, 217–261, 1977.

Model-Theoretic Logics, edited by J. Barwise and S. Feferman, Perspectives in Mathematical Logic, Springer, XVIII + 893 pages, 1985.

James H. Schmerl, *Peano arithmetic and hyper-Ramsey logic*, Transactions of the American Mathematical Society, 296, 2, 481–505, 1986.

Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer, XIV + 445 pages, 1999; 2nd edition, Association for Symbolic Logic, Cambridge University Press, XVI + 444 pages, 2009.

Thank you for your attention!