Non-linear systems of models of two types and some applications

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Forcing with side conditions

Sometimes one wants to add some object O by forcing, but the naive forcing $\mathbb{P}$ for doing so collapses cardinals.

Side conditions have long been used to deal with this problem.

The basic idea (originally due to Todorčević in explicit form) is that we add models N to $\mathbb{P}$-conditions and “force” the condition to be $(N, \mathbb{P})$-generic.

A basic pure side condition forcing: Adding a $\subseteq$-chain covering $[H_\kappa]^{\aleph_0}$ with finite conditions

- Given $\kappa \geq \omega_2$, adding a $\subseteq$-chain

$$\{X_i \,:\, i < \omega_1^V\}$$

to $[H_\kappa]^{\aleph_0}$ by finite approximations will collapse ω_1.

- On the other hand: Let $\mathbb{P}_1$ the forcing of finite $\in$-chains

$$\{M_0, \ldots, M_n\}_\in$$

of countable $M_i \prec H_\kappa$, ordered by $\supseteq$. Then:

(1) $\mathbb{P}_1$ is proper;

(2) if G is $\mathbb{P}_1$-generic, $\bigcup G$ is an $\in$-chain of countable $M \prec H_\kappa^V$ covering H_κ^V;

(3) $\Vdash_{\mathbb{P}_1} |\kappa| = \aleph_1$.

Adding some extra information to $\mathbb{P}_1$ we can get $\mathbb{P}'_1$ so that, for example:

- $\mathbb{P}'_1$ forces that

$$C_{\dot{G}} = \{\delta_M \,:\, M \in p \text{ for some } p \in \dot{G}\}$$

  is a club of ω_1 such that $X \not\subseteq C_{\dot{G}}$ for every infinite set X in V.

  $(\delta_M = \sup(M \cap \omega_1) = M \cap \omega_1.)$

- for some given ladder system

$$\vec{C} = (C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$$

  in V, $\mathbb{P}'_1$ forces $|C_G \cap C_\delta| < \aleph_0$ for every $\delta \in \mathrm{Lim}(\omega_1)$.

Symmetric systems

Suppose we don't want to collapse κ. Suppose, for example, that we want a proper poset dealing with a certain task (e.g. forcing Π_2 consequences of PFA for H_{ω_2}) and forcing $2^{\aleph_0} \geq \kappa$. It may be natural to aim for a forcing with the $\aleph_2$-chain condition.

Here is the corresponding pure side condition:

Definition

Given a predicate $P \subseteq H_\kappa$, a finite set $\mathcal{N}$ of countable models M such that $(M; \in, P \cap M) \prec (H_\kappa; \in, P)$ is a P-*symmetric system* iff:

(1) for all M, $M' \in \mathcal{N}$, if $\delta_M = \delta_{M'}$, then $M \cong M'$ and $\Psi_{M,M'}$ is the identity on $M \cap M'$

($\Psi_{M,M'}$ denotes the unique $\in$-isomorphism $\pi : M \longrightarrow M'$);

(2) for all M, $M' \in \mathcal{N}$, if $\delta_M = \delta_{M'}$, then $\Psi_{M,M'}[\mathcal{N}] \subseteq \mathcal{N}$;

(3) for all M_0, $M_1 \in \mathcal{N}$, if $\delta_{M_0} < \delta_{M_1}$, then there is some $M_1' \in \mathcal{N}$ such that $\delta_{M_1'} = \delta_{M_1}$ and $M_0 \in M_1'$.

(3) is sometimes called *the shoulder axiom*.

Definition

Given a predicate $P \subseteq H_\kappa$, a finite set $\mathcal{N}$ of countable models M such that $(M; \in, P \cap M) \prec (H_\kappa; \in, P)$ is a *P-symmetric system* iff:

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(3) is sometimes called *the shoulder axiom*.

Let $\mathbb{Q}_1$ be the set of all P-symmetric systems ordered by $\supseteq$.

(1) $\mathbb{Q}_1$ is strongly proper: given a countable $M^* \prec H_\theta$ such that κ, $P \in M^*$ and given $\mathcal{N} \in \mathbb{Q} \cap M^*$, $\mathcal{N} \cup \{M\}$ is an $(M^*, \mathbb{Q}_1)$-generic extension of $\mathcal{N}$, where $M = M^* \cap H_\kappa$. In fact, given any $\mathcal{M} \in M^*$ such that $\mathcal{N} \cap M^* \subseteq \mathcal{M}$,

$$\mathcal{A}_M(\mathcal{N}, \mathcal{M}) = \mathcal{N} \cup \bigcup \{\Psi_{M,M'}[\mathcal{M}] \,:\, M' \in \mathcal{N}, \delta_{M'} = \delta_M\}$$

is a common extension of $\mathcal{N}$ and $\mathcal{M}$;

(2) if CH holds, a standard Δ-system argument shows that $\mathbb{Q}_1$ has the $\aleph_2$-c.c. and is in fact $\aleph_2$-Knaster;

(3) if CH holds, $\mathbb{Q}_1$ adds new reals but preserves CH: Let $\mathcal{N} \in \mathbb{Q}_1$ and $\dot{r}_\xi$ a $\mathbb{Q}_1$-name for a real (for $\xi < \omega_2$). Using CH, find $\xi_0 \neq \xi_1$ and sufficiently correct M_{ξ_0}, M_{ξ_1} such that

$$(M_{\xi_0}; \in, P \cap M_{\xi_0}, \dot{r}_{\xi_0}) \cong (M_{\xi_1}; \in, P \cap M_{\xi_1}, \dot{r}_{\xi_1})$$

and $\Psi_{M_{\xi_0}, M_{\xi_1}}$ is the identity on $M_{\xi_0} \cap M_{\xi_1}$. Then

$$\mathcal{N} \cup \{M_{\xi_0}, M_{\xi_1}\}$$

extends $\mathcal{N}$ and forces $\dot{r}_{\xi_0} = \dot{r}_{\xi_1}$.

Two types

One can aim to preserve ω_1 and ω_2, while maybe adding a chain of small models covering H_κ^V. The corresponding notion of side condition, first considered and studied by Neeman, is the following.

Definition

Let $P \subseteq H_\kappa$. Let $\mathcal{S}$ be the class of countable $(M; \in, P \cap M) \prec (H_\kappa; \in, P)$ and $\mathcal{L}$ a class of *sufficiently closed* models $(N; \in, P \cap N) \prec (H_\kappa; \in, P)$ (e.g. internally club or countably closed (${}^\omega N \subseteq N$)). A finite set $\mathcal{C} \subseteq \mathcal{S} \cup \mathcal{L}$ is a *weak P-chain of models of type* $(\mathcal{S}, \mathcal{L})$ iff there is a (unique) linear order $<_\mathcal{C}$ on $\mathcal{C}$ such that:

(1) $Q \in Q'$ if Q' is the immediate $<_\mathcal{C}$-successor of Q;

(2) if $M \in \mathcal{C} \cap \mathcal{S}$, $N \in \mathcal{C} \cap \mathcal{L}$, and $N <_\mathcal{C} M$, then $N \cap M \in \mathcal{C}$ (and $N \cap M < N$);

(3) given $Q < Q'$ in $\mathcal{C}$, if $Q \notin Q'$, then $Q' = M \in \mathcal{S}$ and there is some $N \in \mathcal{C} \cap \mathcal{L}$ such that $N <_\mathcal{C} M$ and $Q <_\mathcal{C} N$.

Note: (2) is equivalent to

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Let $\mathbb{P}_2$ be the set of weak P-chains ordered by $\supseteq$. Then:

(1) $\mathbb{P}_2$ is strongly proper and strongly proper for models in $\mathcal{L}$:

- Given $N^* \prec H_\theta$, $N^* \in \mathcal{L}$, such that κ, $P \in N^*$ and given $\mathcal{C} \in \mathbb{P}_2 \cap N^*$, $\mathcal{C} \cup \{N\}$ is an $(N^*, \mathbb{P}_2)$-generic extension of $\mathcal{C}$, where $N = N^* \cap H_\kappa$. In fact, given any $\mathcal{D} \in N^*$ such that $\mathcal{C} \cap N^* \subseteq \mathcal{D}$,
$$\mathcal{C} \cup \mathcal{D}$$
is a common extension of $\mathcal{C}$ and $\mathcal{D}$;
- Given $M^* \prec H_\theta$, $M^* \in \mathcal{S}$, such that κ, $P \in M^*$ and given $\mathcal{C} \in \mathbb{P}_2 \cap M^*$,
$$\mathcal{C} \cup \{M\} \cup \{N \cap M \,:\, N \in \mathcal{L} \cap M\}$$
is an $(M^*, \mathbb{P}_2)$-generic extension of $\mathcal{C}$, where $M = M^* \cap H_\kappa$. In fact, given any $\mathcal{D} \in M^*$ such that $\mathcal{C} \cap M^* \subseteq \mathcal{D}$,
$$\mathcal{C} \cup \mathcal{D} \cup \{N \cap M' \,:\, M' \in \mathcal{C} \cup \mathcal{D}, N \in (\mathcal{C} \cup \mathcal{D}) \cap \mathcal{L} \cap M'\}$$
is a common extension of $\mathcal{C}$ and $\mathcal{D}$.

(2) if G is $\mathbb{P}_2$-generic, $\bigcup(G \cap \mathcal{L})$ is an $\in$-chain of models $N \in \mathcal{L}$ covering H_κ^V;

(3) $\Vdash_{\mathbb{P}_2} |\kappa| = \aleph_2$.

- Neeman used side conditions of models of two types for example to force PFA with finite conditions (with κ a supercompact cardinal and $\mathcal{L}$ being the class of sufficiently closed $(V_\alpha; \in, F) \prec (V_\kappa; \in, F)$ for a Laver function F for κ).
- There are obstacles to iterating "$\mathbb{P}_2$-proper forcings": otherwise we would for example be able to build models where Club-Guessing on $\omega_2 \cap \mathrm{cf}(\omega)$ (a ZFC-truth) fails.
- Nevertheless, Neeman has built models of high analogues of PFA for certain restricted classes of "$\mathbb{P}_2$-proper" forcings.

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More than two types?

- Veličković proved that the natural version ($\mathbb{P}_3$) of $\mathbb{P}_2$ for models of three types (e.g. countable, ω-closed of size $\aleph_1$ and ω_1-closed of size $\aleph_2$) cannot be proper for both countable models and $\aleph_1$-sized models.
- This argument does not rule out good properties for the pure side condition forcing corresponding to the weaker form of weak chain of three types, where the closure condition (2) is replaced with (2)'.
- There is a framework, due to Veličković, for 'simulating' the presence of models of three types by means of 'virtual models' of models of two types.

Symmetric systems of models of two types

One can naturally try to combine Neeman's notion of weak chain of models of two types with the notion of symmetric system.

One first goal would be to, for example, assume GCH and construct forcings

- dealing with interesting tasks at ω_2 or ω_3,
- which are proper for countable models and (countably closed) $\aleph_1$-models, and
- with the $\aleph_3$-chain condition.

The following is a natural notion of side condition in this context.

In his PhD thesis, Curial Gallart analyzes these side conditions and their use in the construction of models of forcing axioms.

Let $P \subseteq H_\kappa$ and let $\mathcal{S}$ and $\mathcal{L}$ be as in the definition of weak chain of models of two types.

Given $Q \in \mathcal{S} \cup \mathcal{L}$, let $\delta_Q = Q \cap \omega_1$ and $\varepsilon_Q = \sup(Q \cap \omega_2)$.

Let also

$$Q[\omega_1] = \{f(\nu) \,:\, f \in Q \text{ a function}, \nu \in \omega_1\}$$

Given $\mathcal{M} \subseteq \mathcal{S} \cup \mathcal{L}$, let $\mathrm{dom}(\mathcal{M}) = \{\varepsilon_Q \,:\, Q \in \mathcal{M}\}$.

Definition

Let $\mathcal{M} \subseteq \mathcal{S} \cup \mathcal{L}$ be finite. We say that $\mathcal{M}$ is a *P-symmetric system of type* $(\mathcal{S}, \mathcal{L})$ iff:

(1) For any two $Q_0, Q_1 \in \mathcal{M}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then

$$(Q_0[\omega_1]; \in, P \cap Q_0[\omega_1], Q_0) \cong (Q_1[\omega_1]; \in, P \cap Q_1[\omega_1], Q_1)$$

and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(2) If ε_1 is the immediate successor of ε_0 in $\mathrm{dom}(\mathcal{M})$ and $Q_0 \in \mathcal{M}$ is such that $\varepsilon_{Q_0} = \varepsilon_0$, then there is $Q_1 \in \mathcal{M}$ such that $\varepsilon_{Q_1} = \varepsilon_1$ and $Q_0 \in Q_1$.

(3) For all $Q_0, Q_1, Q_1' \in \mathcal{M}$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$,

$$\Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0) \in \mathcal{M}.$$

(4) For every $N \in \mathcal{M} \cap \mathcal{L}$ and every $M \in \mathcal{M} \cap \mathcal{S}$, if $N \in M$, then $N \cap M \in \mathcal{M}$.

The corresponding pure side condition forcing $\mathbb{Q}_2$ (i.e., the set of P-symmetric systems of type $(\mathcal{S}, \mathcal{L})$ ordered by $\supseteq$) has the usual nice properties (with the proof of the properness amalgamation lemma for small models being considerably more involved):

(1) $\mathbb{Q}_2$ is strongly proper for models in $\mathcal{S}$ and for models in $\mathcal{L}$;

(2) if $2^{\aleph_1} = \aleph_2$ holds, a standard Δ-system argument shows that $\mathbb{Q}_2$ has the $\aleph_3$-c.c. and is in fact $\aleph_3$-Knaster;

(3) if $2^{\aleph_1} = \aleph_2$ holds, $\mathbb{Q}_2$ adds new reals but preserves $2^{\aleph_1} = \aleph_2$.

Long strong chains of subsets of ω_1

In the study of higher cardinal characteristics one usually considers spaces such as $[\lambda]^\lambda$ modulo the ideal of subsets of size $<\lambda$.

The space $[\lambda]^\lambda$ modulo the ideal of subsets of size $<\mu$, for some cardinal $\mu < \lambda$, is much harder to deal with.

I will next focus on the space $[\omega_1]^{\aleph_1}$ modulo the ideal of finite sets.

If δ is a limit ordinal, a *strong δ-almost disjoint family of subsets of ω_1* is a sequence $\langle A_\alpha : \alpha < \delta \rangle$ of subsets of ω_1 such that for all $\alpha < \beta < \delta$,

(1) $|A_\alpha| = \aleph_1$, and

(2) $|A_\alpha \cap A_\beta| < \aleph_0$.

- (Baumgartner) If GCH holds and δ is a limit ordinal, there exists a cardinal-preserving poset forcing the existence of a strong δ-almost disjoint family of subsets of ω_1.

Definition

A *strong δ-chain of subsets of ω_1* is a sequence $\langle X_\alpha : \alpha < \delta \rangle$ of subsets of ω_1 such that for all $\alpha < \beta < \delta$,

(1) $|X_\beta \setminus X_\alpha| = \aleph_1$, and

(2) $|X_\alpha \setminus X_\beta| < \aleph_0$.

Definition

If δ is a limit ordinal, a *strong δ-chain of functions from ω_1 to ω_1* is a sequence $\langle f_\alpha : \alpha < \delta \rangle$ of functions such that for all $\alpha < \beta < \delta$,

(1) $f_\alpha \in {}^{\omega_1}\omega_1$, and

(2) $|\{\xi \in \omega_1 : f_\alpha(\xi) \geq f_\beta(\xi)\}| < \aleph_0$.

The existence of a strong chain of functions from ω_1 to ω_1 implies the existence of a strong chain of subsets of ω_1 by identifying each subset of ω_1 with its characteristic function.

Question (Hajnal and Szentmiklóssy): Is it consistent that there exists a strong ω_2-chain of subsets of ω_1? Is it consistent that there exists a strong ω_2-chain of functions from ω_1 to ω_1?

- (Koszmider) Assuming CH, there is a cardinal-preserving poset adding a strong ω_2-chain of functions from ω_1 to ω_1. (This is a forcing with side conditions organized along a morass.)
- Veličković and Venturi obtained a simpler proof of Koszmider's result, by using Neeman-style side conditions to ensure preservation of ω_1 and ω_2.

Inpossibility results

- (Shelah) For every infinite cardinal λ, there cannot be sequences $\langle f_\alpha : \alpha < \lambda^{+3} \rangle$ of functions from ${}^{\lambda^{++}}\lambda^{++}$ increasing modulo the ideal $[\lambda^{++}]^{<\lambda}$.
- (Inamdar) There cannot be sequences $\langle X_\alpha : \alpha < \lambda^{+3} \rangle$ of subsets of λ^{++} increasing modulo the ideal $[\lambda^{++}]^{<\lambda}$. In particular, there cannot be sequences of subsets of ω_2 of length ω_3 increasing modulo finite.

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Theorem

(A.-Gallart) If GCH holds, then there is a cardinal-preserving poset $\mathcal{P}$ forcing the existence of a strong ω_3-chain of subsets of ω_1.

Proof sketch: Assume GCH. Let $\kappa = \omega_3$ and $P = (e_\alpha : \alpha \in \omega_3)$ such that $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each α.
$\mathcal{S}$ is the set of countable $M \prec (H_\kappa; \in, P)$ and $\mathcal{L}$ be the set of $N \prec (H(\omega_3); \in, P)$ such that $|X| = \aleph_1$ and ${}^\omega X \subseteq X$.

A useful fact:

- For all $Q_0, Q_1 \in \mathcal{S} \cup \mathcal{L}$, if

$$(Q_0[\omega_1]; \in, Q_0, P) \cong (Q_1[\omega_1]; \in, Q_1, P),$$

  then $Q_0 \cap Q_1 \cap \omega_3$ is an initial segment of both $Q_0 \cap \omega_3$ and $Q_1 \cap \omega_3$.

Definition

Let $\mathcal{A}$ be a finite subset of $\mathcal{S}$, let $\nu < \omega_1$, and let $\alpha, \beta \in \omega_3$. Then, $\alpha <_{\mathcal{A},\nu} \beta$ holds iff $\alpha < \beta$ and there are $M_0, \ldots, M_n \in \mathcal{A}$ and $\gamma_0 < \cdots < \gamma_{n-1} < \omega_3$ such that

(1) $\sup_{i \leq n} \delta_{M_i} \leq \nu$,

(2) $\alpha \in M_0$ and $\beta \in M_n$, and

(3) $\gamma_i \in M_i \cap M_{i+1} \cap (\alpha, \beta)$, for every $i < n$.

The forcing $\mathcal{P}$ witnessing the theorem is the following.

Conditions in $\mathcal{P}$ are tuples

$$p = (\mathcal{M}_p, \mathcal{A}_p, a_p, d_p, u_p, b_p)$$

such that:

(1) $\mathcal{M}_p$ is an P-symmetric system of type $(\mathcal{S}, \mathcal{L})$.

(2) $\mathcal{A}_p \subseteq \mathcal{M}_p \cap \mathcal{S}$ is such that $N \cap M \in \mathcal{A}_p$ for all $M \in \mathcal{A}_p$ and $N \in \mathcal{M}_p \cap \mathcal{L} \cap M$.

(3) $a_p \in [\omega_3]^{<\omega}$.

(4) $d_p \in [\omega_1]^{<\omega}$.

(5) $u_p = (u_p^\alpha : \alpha \in a_p)$ and for each $\alpha \in a_p$, $u_p^\alpha : d_p \to 2$ is a function.

(6) $b_p = (b_p(\alpha, \beta) : \alpha, \beta \in a_p, \alpha < \beta)$ and for all $\alpha < \beta$ in a_p,

  (a) $b_p(\alpha, \beta) \in [d_p]^{<\omega}$ and

  (b) $b_p(\alpha, \beta) \subseteq \delta_M$ for every $M \in \mathcal{A}_p$ such that $\alpha, \beta \in M$.

(7) For every $\nu \in d_p$ and all $\alpha, \beta \in a_p$, if $\alpha <_{\mathcal{A}_p, \nu} \beta$, then $u_p^\alpha(\nu) \leq u_p^\beta(\nu)$.

(8) For all $\alpha < \beta \in a_p$ and for every $\nu \in d_p \setminus b_p(\alpha, \beta)$, $u_p^\alpha(\nu) \leq u_p^\beta(\nu)$.

Given $\mathcal{P}$-conditions p and q, $q \leq_{\mathcal{P}} p$ iff:

- $\mathcal{M}_q \supseteq \mathcal{M}_p$;
- $\mathcal{A}_q \supseteq \mathcal{A}_p$;
- $a_q \supseteq a_p$;
- $d_q \supseteq d_p$;
- for all $\alpha \in a_p$, $u^\alpha_q \supseteq u^\alpha_p$;
- for all $\alpha < \beta$ in a_p, $b_q(\alpha, \beta) = b_p(\alpha, \beta)$.

We then prove that $\mathcal{P}$

- is proper for models from $\mathcal{S}$ and for models from $\mathcal{L}$,
- is $\aleph_3$-Knaster, and
- adds a strong ω_3-chain $\langle X_i : i < \omega_3 \rangle$ of subsets of ω_1 (by letting $\nu \in X_i$ iff $u^\alpha_p(\xi) = 1$ for some $p \in \dot{G}_{\mathcal{P}}$, where α is the i-member of $\bigcup\{a_p : p \in \dot{G}_{\mathcal{P}}\}$ and ξ is the ν-th member of $\bigcup\{d_p : p \in \dot{G}_{\mathcal{P}}\}$).

  $\square$

Collapsing functions and precipitous ideals

Theorem

(Foreman-Magidor-Shelah) If κ is a supercompact cardinal, then there is a semiproper poset $\mathbb{P} \subseteq V_\kappa$ (thus preserving ω_1) forcing $\omega_2 = \kappa$ and Martin's Maximum, and hence forcing that NS_{ω_1} is saturated.

Theorem

(Foreman-Magidor-Shelah) If κ is a supercompact cardinal and $\mu < \kappa$ is regular, $\mathrm{Coll}(\mu, {<}\kappa)$ forces that NS_μ is precipitous.

Question (Jech): Does any large cardinal notion imply that NS_{ω_1} is precipitous?

F-M-S showed that the answer is No:

Theorem

(Foreman-Magidor-Shelah) If μ is a regular cardinal, then there is a $<\mu$-closed poset $\mathbb{Q} \subseteq 2^\mu$ forcing that NS_μ is not precipitous.

A notable refinement of the first theorem:

Theorem

(Woodin) If δ is a Woodin cardinal, the stationary tower $\mathbb{P}_{<\delta} \subseteq V_\delta$ is such that, if G is $\mathbb{P}_{<\delta}$-generic, then there is, in $V[G]$, an elementary embedding $j : V \longrightarrow M$, ${}^{<\delta}M \cap V[G] \subseteq M$, with critical point ω_1^V (or critical point ω_2^V, ω_3^V, ...).

General question: Assume large cardinals. Is there a *small* forcing $\mathcal{P}$ such that if G is $\mathcal{P}$-generic, then there is an elementary embedding $j : V \longrightarrow M$, M transitive, with $\mathrm{crit}(j) = \omega_1^V$?

Collapsing functions and extender models

Definition

(Schimmerling-Veličković) Given a cardinal κ, $f : \omega_1 \longrightarrow \omega_1$ is a *collapsing function for* κ if there is a club $D \subseteq [H_\kappa]^{\aleph_0}$ such that $\mathrm{ot}(M \cap \kappa) < f(\delta_M)$ for every $M \in D$.

Note: By condensation, in L there is a collapsing function for every κ: just let $f(\alpha)$ be the least β such that $L_\beta \models |\alpha| = \aleph_0$.

Much more is true:

Theorem

(Schimmerling-Veličković) Suppose $L[\vec{E}]$ is a coherent extender model such that every countable model in $L[\vec{E}]$ embedding into some level of $L[\vec{E}]$ is $\omega_1 + 1$-iterable. Then
$L[\vec{E}] \models$ For every $n \geq 2$ there is a collapsing function for $\aleph_n$.

Theorem

(Schimmerling-Veličković) Suppose there is a forcing $\mathcal{P} \subseteq 2^{\aleph_1}$ *adding an elementary embedding* $j : V \longrightarrow M$ *with* M *transitive and* $\mathrm{crit}(j) = \omega_1^V$. *Then*
$V \models$ *There is no collapsing function for* $(2^{\aleph_1})^+$.

Proof.

Let $\kappa = (2^{\aleph_1})^+$. Suppose $F : [H_\kappa]^{<\omega} \longrightarrow H_\kappa$ generates a club $D \subseteq [H_\kappa]^{\aleph_0}$ witnessing that $f : \omega_1 \longrightarrow \omega_1$ is a collapsing function for κ.
Let G be $\mathcal{P}$-generic and let $j : V \longrightarrow M$ be an elementary embedding in $V[G]$ with M transitive and $\mathrm{crit}(j) = \omega_1^V$. Since $|\mathcal{P}| \leq 2^{\aleph_1}$, $j(\omega_1^V) \leq \kappa$.
Let $\alpha = j(f)(\omega_1^V) < j(\omega_1^V) \leq \kappa$ and $X = j[H_\kappa^V]$. Then X is closed under $j(F)$, $X \cap j(\omega_1) = \omega_1^V$, and
$j(f)(X \cap j(\omega_1)) = \alpha < \kappa = \mathrm{ot}(X)$.
By well-foundedness of M, in M there is a set $X \in [j(H_\kappa)]^{\aleph_0}$ closed under $j(F)$, and hence in $j(D)$, such that $X \cap j(\omega_1) = \omega_1^V$, and $j(f)(X \cap j(\omega_1)) = \alpha < \mathrm{ot}(X)$. Contradiction. □

It is easy to force a collapsing function for ω_2 (Wu): forcing with finite symmetric systems $\mathcal{N} \subseteq H_{\omega_2}$ will do (A.-Schindler).

Question

Is there a poset forcing the nonexistence of a small forcing $\mathbb{P}$ *(of size* $2^{\aleph_1}$*) adding an elementary embedding* $j : V^{\mathbb{P}} \longrightarrow M$*,* M *transitive? Is there even a poset forcing the existence of a collapsing function for* $(2^{\aleph_1})^+$*?*

Theorem

(A.-Schindler) Assume GCH below $\aleph_3$ and let $\kappa \geq \aleph_3$ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there is a two-step forcing iteration $\mathcal{P} = \mathcal{P}_0 * \dot{\mathcal{Q}}$ with the following properties.

(1) *$\mathcal{P}_0 \subseteq H_\kappa$ is σ-closed, proper for countably closed $\aleph_1$-sized models, and has the $\aleph_3$-chain condition;*

(2) *$\dot{\mathcal{Q}}$ is a $\mathcal{P}_0$-name for a proper forcing with the $\aleph_2$-chain condition;*

(3) *$\mathcal{P}$ forces $2^{\aleph_1} = \aleph_2$ together with the existence of a collapsing function for κ.*

In particular, $\mathcal{P}$ forces the nonexistence a forcing $\mathbb{P}$ of size $2^{\aleph_1}$ adding an elementary embedding $j : V^{\mathbb{P}} \longrightarrow M$, M transitive.

This theorem improves on an earlier result of Wu:

Theorem

(Wu) There is a set forcing giving rise to a generic extension in which there is no precipitous ideal on ω_1 definable over H_{ω_2}.

Woodin defined the following abstract version of the usual principle of condensation holding in L.

Definition

Suppose M is a transitive set closed under the Gödel operations and $F : \mathrm{Ord} \cap M \to M$ is a bijection. Then F *witnesses Strong Condensation for* M if for any $X \prec (M; \in, F)$, $F_X = F \restriction \mathrm{Ord} \cap M_X$, where M_X denotes the transitive collapse of X and F_X the image of $F \restriction X$ under the collapsing function of X.

Question

Can one always force Strong Condensation for H_{ω_3}*?*

Proof sketch of the theorem

We will use a natural notion of symmetric system of *uncountable* models *with predicates*:

Definition

Let W be a model of $\mathrm{ZFC}^* = \mathrm{ZFC} \setminus \{\text{Power Set}\}$, let $P \subseteq W$ such that $\bigcup P = W$, and let $\mathcal{K}$ be a class of models $N \in W$ of ZFC^* of size $\aleph_1$ such that $\varepsilon_N = \sup(N \cap \omega_2) \in \omega_2$ and $(N; \in, P \cap N) \prec (W; \in, P)$. We call a collection $\mathcal{N} \subseteq W$ a *P-symmetric system of models in $\mathcal{K}$ with predicates* if $\mathcal{N}$ is a set of ordered pairs (N, S), where $N \in \mathcal{K}$ and $S \subseteq N$, such that for all (N_0, S_0), $(N_1, S_1) \in \mathcal{N}$:

(1) If $\varepsilon_{N_0} = \varepsilon_{N_1}$, then
  - (a) $(N_0; \in, S_0, P \cap N_0)$ and $(N_1; \in, S_1, P \cap N_1)$ are isomorphic;
  - (b) Ψ_{N_0, N_1} is the identity on $N_0 \cap N_1$;
  - (c) $\Psi_{N_0, N_1}[\mathcal{N}] \subseteq \mathcal{N}$.

(2) If $\varepsilon_{N_0} < \varepsilon_{N_1}$, then there is some $(N_1', S_1') \in \mathcal{N}$ such that $\varepsilon_{N_1'} = \varepsilon_{N_1}$ and $(N_0, S_0) \in N_1'$.

Definition

Let W be a model of ZFC^* and let $P \subseteq W$ such that $\bigcup P = W$. We call a graph $\mathcal{G} \subseteq W$ a *weakly P-symmetric graph of countable models* if $V(\mathcal{G})$ is a finite set of elementary submodels of $(W; \in, P)$ and the following holds for all M_0, $M_1 \in V(\mathcal{G})$.

(1) If $\{M_0, M_1\} \in E(\mathcal{G})$, then $M_0 \in M_1$ or $M_1 \in M_0$.

(2) If $\delta_{M_0} = \delta_{M_1}$, then
  - (a) $(M_0; \in, P \cap M_0)$ and $(M_1; \in, P \cap M_1)$ are isomorphic;
  - (b) Ψ_{M_0, M_1} is the identity on M for every $M \in M_0 \cap M_1$ such that $\{M, M_0\} \in E(\mathcal{G})$ and $\{M, M_1\} \in E(\mathcal{G})$.

(3) For every two vertices M_0, M_1 in $V(\mathcal{G})$, if $\delta_{M_0} < \delta_{M_1}$, then there is some $M_1' \in V(\mathcal{G})$ with $\delta_{M_1'} = \delta_{M_1}$ such that $\{M_0, M_1'\} \in E(\mathcal{G})$ and $\{M, M_1'\} \in E(\mathcal{G})$ for every M such that $\{M, M_0\}_{\in} \in E(\mathcal{G})$.

The motivation to use the present weak form of symmetry in (2)(b) is to dodge the following obstacle to forcing $2^{\aleph_1} = \aleph_2$ while covering H_κ with a collection of models enjoying the stronger form of symmetry where we require Ψ_{M_0,M_1} to be the identity on $M_0 \cap M_1$.

Fact

Let W be a transitive model of $\mathsf{ZFC}^* + CH$ and suppose $\mathcal{M}$ is a collection of countable members of W such that $[\kappa]^2 \subseteq \bigcup \mathcal{M}$, where $\kappa = W \cap Ord$. Suppose Ψ_{M_0,M_1} is the identity on $M_0 \cap M_1$ whenever M_0 and M_1 are isomorphic members of $\mathcal{M}$. Then $2^{\aleph_1} \geq \kappa$.

Proof.

For every $\alpha < \kappa$, let $r_\alpha \subseteq \omega_1$ encode the set X_α of $(\tau, \nu) \in \omega_1 \times \omega_1$ where, for some $M \in \mathcal{M}$ such that $\alpha \in M$, τ is the isomorphism type of M and $\nu = \mathrm{ot}(M \cap \alpha)$.

Then $r_\alpha \neq r_\beta$ for all $\alpha \neq \beta$ in κ. For this, let $M \in \mathcal{M}$ be such that $\alpha, \beta \in M$. Let τ be the isomorphism type of M and ν the order type of $M \cap \beta$. Then $(\tau, \nu) \in X_\beta$, as witnessed by M, but $(\tau, \nu) \notin X_\alpha$: if $M' \in \mathcal{M}$ with $\alpha \in M'$, $M' \cong M$ and $\mathrm{ot}(M' \cap \alpha) = \nu$, then $\Psi_{M',M}(\alpha) = \beta \neq \alpha$ although $\alpha \in M \cap M'$, which contradicts our hypotheses. □

The present notion of weakly P-symmetric graph of countable models allows for a suitable amalgamation lemma:

- If $\mathcal{G}$ is such a graph, $M \in V(\mathcal{G})$ and $\mathcal{H} \in M$ is a P-symmetric graph which relates nicely to the part of $\mathcal{G}$ that M can see, then after adding some carefully chosen vertices and edges to

$$(V(\mathcal{G}) \cup V(\mathcal{H}), E(\mathcal{G}) \cup E(\mathcal{H}))$$

we obtain a P-symmetric graph extending $\mathcal{G}$ and $\mathcal{H}$.

Proof of the theorem:

Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ and let $\kappa \geq \aleph_3$ a regular cardinal such that $2^{<\kappa} = \kappa$.

Let $\mathcal{K}$ be the class of countably closed elementary $N \prec H_\theta$ for some θ such that $|N| = \aleph_1$.

Our first forcing $\mathcal{P}_0$ is the collection, ordered under $\supseteq$, of all countable H_κ-symmetric $\mathcal{K}$-systems with predicates.

- $\mathcal{P}_0$ is σ-closed;
- $\mathcal{P}_0$ is proper for models from $\mathcal{K}$;
- $\mathcal{P}_0$ has the $\aleph_3$-c.c.
- $\mathcal{P}_0$ preserves $2^{\aleph_1} = \aleph_2$.

Let G_0 be $\mathcal{P}_0$-generic. Then $\mathcal{N}^{G_0} = \bigcup G_0$ is a H_κ^V-symmetric system of models in $\mathcal{K}$ with predicates.

Let us next work in $V[G_0]$.

Our forcing $\mathcal{Q}$ will be the collection of all ordered pairs $q = (\mathcal{N}_q, \mathcal{G}_q)$, where:

(1) $\mathcal{N}_q$ is a finite subset of

$$\{(N, \mathrm{Sat} \cap N) \in \mathcal{N}^{G_0} : (N; \in, \mathrm{Sat} \cap N) \prec (H^V_\kappa; \in, \mathrm{Sat})\};$$

(2) $\mathcal{G}_q$ is a weakly P-symmetric system of countable models;

(3) for all $(N_0, S_0), (N_1, S_1) \in \mathcal{N}_q$, if $\epsilon_{N_0} = \epsilon_{N_1}$, then

 (a) $\Psi_{N_0,N_1}[V(\mathcal{G}_q)] \subseteq V(\mathcal{G}_q)$,

 (b) $\Psi_{N_0,N_1}[E(\mathcal{G}_q)] \subseteq E(\mathcal{G}_q)$, and

 (c) $\Psi_{N_0,N_1}[\mathcal{N}_q] \subseteq \mathcal{N}_q$.

Given q_0, $q_1 \in \mathcal{Q}$, q_1 extends q_0 iff

- $\mathcal{N}_{q_0} \subseteq \mathcal{N}_{q_1}$,
- $V(\mathcal{G}_{q_0}) \subseteq V(\mathcal{G}_{q_1})$, and
- $E(\mathcal{G}_{q_0}) \subseteq E(\mathcal{G}_{q_1})$.

We then have:

- $\mathcal{Q}$ is proper for countable models and models from $\mathcal{N}^{G_0}$;
- $\mathcal{Q}$ has, in fact, the $\aleph_2$-c.c.;
- $\mathcal{Q}$ preserves $2^{\aleph_1} = \aleph_2$.

Let now H be $\mathcal{Q}$-generic over $V[G_0]$ and let us work in $V[G_0][H]$. Let

$$\mathcal{V}^H = \bigcup\{V(\mathcal{G}_q) \,:\, q \in H\}$$

and

$$\mathcal{E}^H = \bigcup\{E(\mathcal{G}_q) \,:\, q \in H\}$$

Let $h : \omega_1 \to \omega_1$ be given by

$$h(\delta) = \mathrm{ot}(M \cap \kappa) + 1$$

for some (equivalently, any) $M \in \mathcal{V}^H$ with δ_M being the minimum of

$$\{\delta_{M'} \,:\, M' \in \mathcal{H}^H\} \setminus \delta$$

The following lemma shows that h is a collapsing function for κ and thus completes the proof of the theorem.

Lemma

There is a club D of $[H_\kappa^{V[G_0][H]}]^{\aleph_0}$ in $V[G_0][H]$ such that $\mathrm{ot}(M \cap \kappa) < h(\delta_M)$ for every $M \in D$.

Proof: Let θ large enough cardinal and let us work in $V[G_0][H]$. Enough to show that for every countable $M^* \prec H_\theta^{V[G_0][H]}$ with $\mathcal{E}^H \in M^*$, $\mathrm{ot}(M^* \cap \kappa) < h(\delta_{M^*})$.

Let $M = M^* \cap H_\kappa^V$ and $\delta = \delta_{M^*}$. By correctness of M^* and since $\mathcal{E}^H$ is in this model, we may find an $\in$-chain $(M_n)_{n<\omega}$ of members of $\mathcal{V}^H \cap M^*$ such that

- $M = \bigcup_n M_n$ and
- $\{M_n, M_m\} \in \mathcal{E}^H$ for all $n < m$.

Let

$$\delta' = \min(\{\delta_{M'} : M' \in \mathcal{M}^H\} \setminus \delta)$$

For each $n < \omega$, by the version of the shoulder axiom for $(\mathcal{V}^H, \mathcal{E}^H)$ there is $M'_n \in \mathcal{V}^H$ such that $\delta_{M'_n} = \delta'$ and $\{M_k, M'_n\} \in \mathcal{E}^H$ for all $k \leq n$.
Let M' be the direct limit of the system

$$\langle M'_n, \Psi_{M'_n, M'_m} : n \leq m < \omega \rangle$$

and, for each n, let $\pi_n : M'_n \to M'$ be the corresponding limit map. Since each $\Psi_{M'_n, M'_m}$ is an isomorphism between M'_n and M'_m, it follows that $\pi_n : M'_n \to M'$ is an isomorphism for each n.
Let now $i : M \cap \kappa \to M'$ be the function sending α to $\pi_n(\alpha)$ if $n < \omega$ is least such that $\alpha \in M_n$.

By an instance of clause (2)(b) in Definition of weakly symmetric graph, $\Psi_{M'_n, M'_m}$ is the identity on M_n for all $m \geq n$ as $\{M_n, M'_n\}$ and $\{M_n, M'_m\}$ are both in $\mathcal{E}^H$.

It follows that i is an order-preserving function into $M' \cap \kappa$: for all $\alpha < \beta$ in $M \cap \kappa$, if n_0 is least such that $\alpha \in M_{n_0}$, n_1 is least such that $\beta \in M_{n_1}$, and $n = \max\{n_0, n_1\}$, then

$$i(\alpha) = \pi_{n_0}(\alpha) = (\pi_n \circ \Psi_{M_{n_0}, M_n})(\alpha) = \pi_n(\alpha)$$

and

$$\pi_n(\alpha) < \pi_n(\beta) = (\pi_n \circ \Psi_{M_{n_1}, M_n})(\beta) = \pi_{n_1}(\beta) = i(\beta)$$

But then

$$\mathrm{ot}(M^* \cap \kappa) = \mathrm{ot}(M \cap \kappa) \leq \mathrm{ot}(M' \cap \kappa) = \mathrm{ot}(M'_0 \cap \kappa) < h(\delta),$$

where the last inequality holds by the definition of $h(\delta)$ and the choice of δ' since $\delta_{M'_0} = \delta'$.

□

Note: Forcing with finite symmetric systems of countable models ($\mathbb{Q}_1$) gives a collapsing function for κ by the same argument as above.

However, $\mathbb{Q}_1$ will force $2^{\aleph_1} = \kappa > \aleph_2$.

It is because we need to preserve $2^{\aleph_1} = \aleph_2$ that we use $\mathcal{N}^{G_0}$, and so on.

One could have hoped to, instead of $\mathcal{P}_0 * \dot{\mathcal{Q}}$, force with a variant of P-symmetric systems of models of type $(\mathcal{S}, \mathcal{L})$ where one is required to extend intersections $N \cap M$ to a model M' isomorphic to M and such that $\Psi_{M,M'}$ is the identity on $M \cap M'$. But then we cannot guarantee the shoulder axiom when amalgamating conditions.

And we know this approach cannot work because of the obstacle we saw:

Fact

Let W be a transitive model of $\mathrm{ZFC}^* + CH$ and suppose $\mathcal{M}$ is a collection of countable members of W such that $[\kappa]^2 \subseteq \bigcup \mathcal{M}$, where $\kappa = W \cap Ord$. Suppose Ψ_{M_0,M_1} is the identity on $M_0 \cap M_1$ whenever M_0 and M_1 are isomorphic members of $\mathcal{M}$. Then $2^{\aleph_1} \geq \kappa$.