On the Π^1_2 consequences of Π^1_1-CA_0

Yudai Suzuki (Joint work with Keita Yokoyama)

MOPA

April 18 (in NY), April 19 (in Japan), 2024

Summery

In this talk, we

- introduce a sequence of theories $T_0, T_1, T_2, \ldots$ such that

  $T_0 < T_1 < T_2 \cdots$ and
  $\mathsf{ACA}_0 + \bigcup_n T_n = \{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$,

  where
  - $T < S$ and $T = S$ mean $\mathrm{Thm}(T) \subsetneq \mathrm{Thm}(S)$ and $\mathrm{Thm}(T) = \mathrm{Thm}(S)$ respectively,
- compare this sequence and a weaker variant of Σ^0_n Ramsey's theorem or $(\Sigma^0_1)_n$ determinacy.

Motivation

There is a very large gap between ATR_0 and $\Pi^1_1\text{-}\mathsf{CA}_0$:

Fact

Any consistent Π^1_2 extension of ATR_0 does not prove $\Pi^1_1\text{-}\mathsf{CA}_0$.

There are some theorems such that

- that are represented by a Π^1_2 sentence, but
- there are only a few results on the Π^1_2 upper bound for them.

  Menger's theorem in graph theory, Nash-Williams' theorem in bqo theory and Kruskal's theorem for trees...

Question : What is the structure of $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$?

In this section, we introduce an increasing sequence of theories $\langle T_i \rangle_i$ slicing $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$.

Definition

Let X be a set. The hyperjump $\mathsf{HJ}(X)$ of X is the set of all indices of X-computable well-ordering.

Fact

Any $\Pi^{1,X}_1$ set is many-one reducible to $\mathsf{HJ}(X)$.

Definition

Π^1_1-CA_0 is ACA_0 plus $\forall X \exists Y (Y = \mathsf{HJ}(X))$.

Π^1_1- CA_0 is ACA_0 plus $\forall X \exists Y (Y = \mathsf{HJ}(X))$.

Observation

$$\begin{aligned}\Pi^1_1\text{-}\,\mathsf{CA}_0 &= \mathsf{ACA}_0 + \forall X \exists Y (Y = \mathsf{HJ}(X)) \\ &= \mathsf{ACA}_0 + \forall X \exists Y (Y = \mathsf{HJ}^2(X)) \\ &= \mathsf{ACA}_0 + \forall X \exists Y (Y = \mathsf{HJ}^3(X)) \\ &= \cdots\end{aligned}$$

Here, $\mathsf{HJ}^1(X) = \mathsf{HJ}(X)$ and $\mathsf{HJ}^{n+1}(X) = \mathsf{HJ}(\mathsf{HJ}^n(X))$.

Theorems provable from Π^1_1- CA_0 should be classified into HJ-level,HJ^2-level,HJ^3-level,...

Theorem

Let $\theta(X, Y, Z)$ be Σ^1_0 such that

$$\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \forall X \exists Y \forall Z \theta(X, Y, Z).$$

Then, there exists $n \in \omega$ such that

$$\mathsf{ACA}_0 \vdash \forall X, W (W = \mathsf{HJ}^n(X) \to \exists Y \leq_{\mathrm{T}} W \forall Z \theta(X, Y, Z)))$$

[Proof] Rewrite Montalban and Shore's work in [MoS] carefully.

Corollary (classification according to HJ^n-level)

For any Π^1_2 sentence σ such that $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma$, there exists the smallest n_σ satisfying the above condition.

- The statement $\forall X \exists Y (Y = \mathsf{HJ}^n(X))$ is Π^1_3.
- We mainly focus on Π^1_2 statements.
- We would like to introduce a nice Π^1_2 variant of $\forall X \exists Y (Y = \mathsf{HJ}^n(X))$.
  - $\rightarrow$ We use coded ω-models.

Coded ω-model: A structure of second-order arithmetic

- its first order part is $\mathbb{N}$ (the same as the ground model)
- its second order part is coded by a set.

Let $\mathcal{M} = \langle \mathcal{M}_i \rangle_i$ be a coded ω-model. Then, for any sentence σ with parameters from $\mathcal{M}$, the condition

$$\mathcal{M} \models \sigma$$

is a Σ^1_1 condition.
Therefore, a formula of the form

$$\forall X \exists \mathcal{M}\text{: coded } \omega\text{-model}(X \in \mathcal{M} \models \sigma)$$

is Π^1_2.

$\beta^1_0\mathsf{RFN}(n)$

$$\forall X \exists \mathcal{M}\text{: coded } \omega\text{-model}$$
$$(X \in \mathcal{M} \models \mathsf{ACA}_0 + \exists Y (Y = \mathsf{HJ}^n(X))).$$

- Each $\beta^1_0\mathsf{RFN}(n)$ is a Π^1_2 sentence.
- Each $\beta^1_0\mathsf{RFN}(n)$ is provable from $\Pi^1_1\text{-}\mathsf{CA}_0$.
- Over ACA_0, $\beta^1_0\mathsf{RFN}(0) < \beta^1_0\mathsf{RFN}(1) < \beta^1_0\mathsf{RFN}(2) < \cdots$.
- Each $\sigma \in \Pi^1_2$ such that $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma$ is provable from $\mathsf{ACA}_0 + \beta^1_0\mathsf{RFN}(n)$ for some n.

The theories $\mathsf{ACA}_0 + \beta^1_0\mathsf{RFN}(0), \mathsf{ACA}_0 + \beta^1_0\mathsf{RFN}(1), \ldots$ slice the set $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$.

The name $\beta^1_0\mathsf{RFN}$ comes from the notion of β-models :

Definition

Let $\mathcal{M}$ be a coded ω-model. We say $\mathcal{M}$ is a β-model if it is Σ^1_1 absolute in the sense that for any Σ^1_1 formula $\theta(\vec{X})$

$$\forall \vec{A} \in \mathcal{M}(\theta(\vec{A}) \leftrightarrow \mathcal{M} \models \theta(\vec{A}))$$

Remark :

- Coded ω-models are Σ^1_0 absolute.
- $[\mathcal{M} \models \theta(\vec{A})] \to \theta(\vec{A})$ is trivial.

Fact

Over ACA_0, for any X, TFAE.

- $\mathsf{HJ}(X)$ exists.
- A coded β-model $\mathcal{M}$ such that $X \in \mathcal{M}$ exists.

Therefore, $\beta^1_0\mathsf{RFN}(n)$ is equivalent to

$$\forall X \exists \mathcal{M}_0, \dots, \mathcal{M}_n : \text{coded } \omega\text{-models}(X \in \mathcal{M}_0 \in_\beta \cdots \in_\beta \mathcal{M}_n \models \mathsf{ACA}_0).$$

Here $\mathcal{M}_i \in_\beta \mathcal{M}_{i+1}$ means that $\mathcal{M}_i \in \mathcal{M}_{i+1}$ and $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ are Σ^1_1 absolute.

Remark

- The superscript 1 is from [$\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ are Σ^1_1 absolute].
- The subscript 0 is from [$\mathcal{M}_n$ and the ground model are Σ^1_0 absolute].

In this section, we compare the $\beta^1_0\mathsf{RFN}$-hierarchy and Menger's theorem, Nash-Williams' theorem and Kruskal's theorem.

- Menger's theorem, Nash-Williams' theorem and Kruskal's theorem are Π^1_2 statements provable from $\Pi^1_1\text{-}\mathsf{CA}_0$.
- H. Towsner gave a Π^1_2 upper bound.
- He used a variant of leftmost path principle.

Leftmost Path Principle

TFAE over ACA_0:

- $\Pi^1_1\text{-}\mathsf{CA}_0$,
- any ill-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a leftmost path (w.r.t. the lexicographical order).

Towsner introduced a restricted *leftmost-ness*.

Let T be a tree, $f \in [T]$ and α be a well-order.

The path f is a Δ^0_α-leftmost path if

$$\forall g \in [T](g \leq_T f^{(\alpha)} \to f \leq_l g).$$

The path f is leftmost in $[T] \cap \Delta^{0,f}_\alpha$.

- $\Delta^0_\alpha\mathsf{LPP}$: any ill-founded tree has a Δ^0_α-leftmost path.
- TLPP: $\forall\alpha(\mathrm{WO}(\alpha) \to \Delta^0_\alpha\mathsf{LPP})$.

Theorem (Towsner)

- $\mathsf{ATR}_0 \leq \Delta^0_0\mathsf{LPP} < \mathsf{TLPP}$ and
- MT and NWT are provable from TLPP,
- KT is provable from $\Delta^0_2\mathsf{LPP}$.

How strong are $\Delta^0_2\mathsf{LPP}$ and TLPP in $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$?

Theorem

Over ACA_0,

$$\mathsf{ATR}_0 < \Delta^0_2\mathsf{LPP} < \beta^1_0\mathsf{RFN}(1) = \mathsf{ALPP} < \mathsf{TLPP} < \beta^1_0\mathsf{RFN}(2).$$

Here, ALPP is a variant of TLPP.

- Some fragments of Galvin-Prikry's theorem ($\Sigma^0_2\mathsf{Ram}, \Sigma^0_3\mathsf{Ram}\ldots$) are equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$.
  - The hierarchy of $\Sigma^0_n\mathsf{Ram}$ collapses in reverse math.
- From the viewpoint of computability, $\Sigma^0_2\mathsf{Ram}, \Sigma^0_3\mathsf{Ram}\ldots$ should be separated.
- Comparing $\beta^1_0\mathsf{RFN}$-hierarchy and $\Sigma^0_n\mathsf{Ram}$-hierarchy, we can say the latter is strict in a sense.

Definitions :

- $[X]^{\mathbb{N}}$: the set of infinite subsets of X.
- $\mathcal{A} \subseteq [\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property if there exists a set $H \in [\mathbb{N}]^{\mathbb{N}}$ such that

$$[H]^{\mathbb{N}} \subseteq \mathcal{A} \vee [H]^{\mathbb{N}} \cap \mathcal{A} = \varnothing.$$

Fact

Over ACA_0, TFAE.

- Π^1_1-CA_0.
- Any Σ^0_2 definable class has the Ramsey property ($\Sigma^0_2\mathsf{Ram}$).
- Any Σ^1_0 definable class has the Ramsey property ($\Sigma^1_0\mathsf{Ram}$).

Definition (arithmetical reduction)

Let X, Y be sets. We write $X \leq_{\mathrm{T}}^{\mathrm{a}} Y$ if $\exists n(X \leq_{\mathrm{T}} Y^{(n)})$.

We say $\mathcal{A}$ has the relative Ramsey property if there exists $H \in [\mathbb{N}]^{\mathbb{N}}$ such that

$$([H]^{\mathbb{N}} \cap \{G : G \leq_{\mathrm{T}}^{\mathrm{a}} H\}) \subseteq \mathcal{A} \vee$$
$$([H]^{\mathbb{N}} \cap \{G : G \leq_{\mathrm{T}}^{\mathrm{a}} H\}) \cap \mathcal{A} = \varnothing.$$

Definition (rel(Σ^i_jRam))

Any Σ^i_j definable class has the relative Ramsey property.

Theorem

Over ACA_0,

- $\beta^1_0\mathsf{RFN}(n)$ proves $\mathrm{rel}(\Sigma^0_n\mathsf{Ram})$ for $n > 0$.
- $\mathrm{rel}(\Sigma^0_{2n}\mathsf{Ram})$ proves $\beta\mathsf{RFN}(n)$ for $n > 0$.

Although the $\Sigma^0_n\mathsf{Ram}$ hierarchy collapses, the $\mathrm{rel}(\Sigma^0_n\mathsf{Ram})$ hierarchy does not.

Remark

This result reflects a computability theoretic intuition that [a homogeneous set for $(\Sigma^0_n\mathsf{Ram})$ is computable from the n-th hyperjump].

Remark

The last result probably be improved as follows: Over ACA_0,

$$\beta^1_0\mathsf{RFN}(n) \leftrightarrow \mathrm{rel}(\Sigma^0_n).$$

(Cf. A. Marcone and G. Marco, "The Galvin-Prikry theorem in the Weihrauch lattice", to appear.)

In this section, we compare a restricted variant of the determinacy for Gale-Stewart game and $\beta^1_0\mathsf{RFN}$-hierarchy

Let $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$. Consider the following two players game $G_{\mathcal{A}}$:

1. Let $n = 0$.
2. Player 0 plays $x_{2n} \in \mathbb{N}$.
3. Player 1 plays $x_{2n+1} \in \mathbb{N}$.
4. Increment n and return to step 2.
5. After ω-steps this game yields a sequence $(x_i)_i \in \mathbb{N}^{\mathbb{N}}$.

Player 0 wins $G_{\mathcal{A}}$ iff $(x_i)_i \in \mathcal{A}$.

Definitions :

- A function $S : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is called a strategy.
- For strategies S, S', a sequence $(x_i)_i$ is the play along S, S' ($(x_i)_i = S \otimes S'$) if

$$x_{2i} = S(x_0, \ldots, x_{2i-1}) \text{ and } x_{2i+1} = S'(x_0, \ldots, x_{2i}).$$

- A strategy S is a winning strategy for 0 in $G_{\mathcal{A}}$ if for any strategy S', $S \otimes S' \in \mathcal{A}$.
- A strategy S' is a winning stratgy for 1 in $G_{\mathcal{A}}$ if for any strategy S, $S \otimes S' \notin \mathcal{A}$.
- $\mathcal{A}$ is determined if $G_{\mathcal{A}}$ has a winning strategy.

Γ-Det

Any class $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ defined by a $\varphi \in \Gamma$ is determined.

Fact

Over ACA_0, TFAE.

- Π^1_1- CA_0.
- $(\Sigma^0_1)_2$-Det.
- $(\Sigma^0_1)_n$-Det for $n > 2$.

Here, $(\Sigma^0_1)_n$ is the class defined by

- $(\Sigma^0_1)_1$ is just Σ^0_1.
- $(\Sigma^0_1)_{n+1}$ is of the form $\varphi \wedge (\neg\psi)$ where φ is Σ^0_1 and ψ is $(\Sigma^0_1)_n$.

Let $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$.

- A strategy S is a relative-winning strategy for player 0 if $\forall S' \leq^{\mathrm{a}}_{\mathrm{T}} S(S \otimes S' \in \mathcal{A})$.
- A strategy S' is a relative-winning strategy for player 1 if $\forall S \leq^{\mathrm{a}}_{\mathrm{T}} S'(S \otimes S' \notin \mathcal{A})$.
- $\mathcal{A}$ is pseudo-determined if the game $G_{\mathcal{A}}$ has a pseudo-winning strategy.

rel($(\Sigma^0_1)_n$-Det)

Any $(\Sigma^0_1)_n$ definable class is pseudo-determined.

Theorem

Over ACA_0,

- $\beta^1_0\mathsf{RFN}(1) < \mathrm{rel}((\Sigma^0_1)_2\mathsf{Det}) < \beta^1_0\mathsf{RFN}(2)$.
- $\mathrm{rel}((\Sigma^0_1)_n\text{-}\mathsf{Det}) \leq \beta^1_0\mathsf{RFN}(n)$.
- $\beta^1_0\mathsf{RFN}(n) < \mathrm{rel}((\Sigma^0_1)_{p(n)}\text{-}\mathsf{Det})$.

Here, $p(n)$ is a certain elementary function.

Question

Is $\beta^1_0\mathsf{RFN}(n)$ provable from $\mathrm{rel}((\Sigma^0_1)_{n+1}\mathsf{Det})$ over ACA_0?

- The Π^1_2 consequences of Π^1_1- CA_0 is covered by the hierarchy $\beta^1_0\mathsf{RFN}(0) < \beta^1_0\mathsf{RFN}(1) < \beta^1_0\mathsf{RFN}(2) < \cdots$.
- Some existing Π^1_2 consequences (MT, NWT and KT) are bounded by $\beta^1_0\mathsf{RFN}(2)$.
- The hierarchy of the relative Ramsey's theorem/determinacy also cover $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{- }\mathsf{CA}_0 \vdash \sigma\}$.

References

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