# Independence in PA: The Method of $(\mathcal{L}, n)$-Models

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# Introduction

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In arithmetic the project of finding a non-logical example of independence was achieved by Paris and Harrington in 1977 (more on this later). While there has been extensive research into "mathematical independence" most known examples resemble Paris and Harrington's original example.

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Reflecting on this discussion Harrington (and others) asked whether there were true, unprovable $\Pi^0_1$ "mathematical" statements. Note that this is the lowest possible complexity since any true $\Sigma^0_1$ statement is provable in PA. Note also that $\mathrm{Con}(\mathsf{PA})$ is $\Pi^0_1$.

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Shelah's paper, which contains much more than just the true unprovable $\Pi^0_1$ sentence including an alternative proof of the Paris-Harrington theorem, is a wealth of interesting results in models of PA. However, for whatever reason, it seems to have never been fully digested by the community.

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- Use the theory developed to present Shelah's strikingly simple alternative proof of the Paris-Harrington Theorem
- Use $(\mathcal{L}, n)$-models to introduce a new true $\Pi^0_1$ sentence and show that it is not provable in PA

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When we discuss infinite objects, to avoid awkwardness we may also argue in $\mathsf{ACA}_0$ since this theory is conservative over PA for first order sentences. This strategy was suggested by the anonymous referee, thank you if you are here!

I now want to develop the theory of a certain class of sequences of finite structures called $(\mathcal{L}, n)$-models. Most of this section is essentially due to Shelah, however my presentation here is more formal and systematized than in his paper. Moreover, in anticipation of later applications we will often present stronger versions of his original ideas.

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The essence of this idea also appears in work of Kripke (unpublished) and has been mentioned in expositions of his work by Putnam and Quinsey.

# $(\mathcal{L}, n)$-Models

Let's fix a finite signature first order language $\mathcal{L}$ extending $\mathcal{L}_{\mathrm{PA}}$. A *partial $\mathcal{L}$-structure* is a set with interpretations for constants, relations and function symbols from $\mathcal{L}$ defined on it in the usual way except that functions can be partial.

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Let $n \in \omega$ (possibly non standard) the structure $\mathcal{M}_n$ is the structure whose universe is $n = \{0, 1, ..., n-1\}$ with $+$, $\times$ etc defined as usual but restricted to this set.

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If $\mathcal{M}$ is a partial $\mathcal{L}$-structure, $a \in \mathcal{M}$ and $f$ is a function symbol in $\mathcal{L}$ so that $f^{\mathcal{M}}(a)$ is not defined we treat any formula containing the string "$f(a)$" as syntactic nonsense.

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If $\mathcal{M}$ is a partial $\mathcal{L}$-structure, $a \in \mathcal{M}$ and $f$ is a function symbol in $\mathcal{L}$ so that $f^{\mathcal{M}}(a)$ is not defined we treat any formula containing the string "$f(a)$" as syntactic nonsense. For instance, $\mathcal{M}_6 \models 1 + 1 = 2$ but the term "$3 \times 4$" doesn't appear in any sentence $\mathcal{M}_6$ models.

# $(\mathcal{L}, n)$-Models

Given two partial $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$ in the normal sense and for each function symbol $f$ in $\mathcal{L}$ or arity $k$ (say), $f^{\mathcal{B}} \upharpoonright [\mathcal{A}]^k$ is total. In other words, $\mathcal{B}$ closes functions under $\mathcal{A}$.

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**Example (Key Example Continued)**

Let $\mathcal{L}$ be as before and let $n > m^2$. Then $\mathcal{M}_m \subseteq \mathcal{M}_n$ since for all $k, l < m$, $kl < n$.

# $(\mathcal{L}, n)$-Models

Fix a natural number $n$. The following definition is the main character of the talk.

## Definition ($(\mathcal{L}, n)$-Model)

An $(\mathcal{L}, n)$-*model* $\vec{\mathcal{A}} = \langle \mathcal{A}_0, ..., \mathcal{A}_{n-1} \rangle$ is a sequence of partial $\mathcal{L}$-structures of length $n$ so that for all $i < n - 1$ $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$.

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Given an $(\mathcal{L}, n)$-model $\vec{\mathcal{A}}$, I denote by $\vec{\mathcal{A}}^{[i,j]}$ the sequence $\mathcal{A}_i \subseteq \mathcal{A}_{i+1} \subseteq ... \subseteq \mathcal{A}_j$. Note that this is an $(\mathcal{L}, j - i + 1)$-model.

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## Example (Key Example Continued Again)

Let $\vec{m} = m_0 < m_1 < ... < m_{n-1}$ be a sequence of natural numbers so that for all $i < n - 1$, $m_i^2 < m_{i+1}$. The associated $(\mathcal{L}, n)$-model is $\vec{M}_{\vec{m}} = \langle \mathcal{M}_{m_0}, ..., \mathcal{M}_{m_{n-1}} \rangle$. We call such a model *square increasing*.

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Let's set some notation and terminology. Given an $(\mathcal{L}, n)$-model $\vec{\mathcal{A}}$ I will always write $\mathcal{A}_i$ for the $i^{\text{th}}$ model in the sequence. I sometimes refer to $\mathcal{A}_i$ as the $i^{\text{th}}$ *model of* $\vec{\mathcal{A}}$ and in particular we call $\mathcal{A}_{n-1}$ the *top model*.

The point is that the $(\mathcal{L}, n)$-models satisfy a kind of satisfaction relation called *fulfillment* which can be used to code consistency statements into finite combinatorial ones.

From now on, given a formula $\varphi$, denote by $dp(\varphi)$ the *depth* of $\varphi$ i.e. the number of quantifiers appearing in $\varphi$ (NOT the number of quantifier alternations). Denote by $|\varphi|$ the syntactic length of $\varphi$.

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Let $\varphi(\vec{x})$ be an $\mathcal{L}$ formula, $\vec{\mathcal{A}}$ an $(\mathcal{L}, n)$-model from some $n$ and and $\vec{a}$ a tuple of elements of the same arity as $\vec{x}$ from $\mathcal{A}_{n-1}$ (the top model).

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- If $\varphi := \psi_1 \wedge \psi_2$, then $\vec{\mathcal{A}} \models^* \varphi(\vec{a})$ if and only if $\vec{\mathcal{A}} \models^* \psi_1(\vec{a})$ and $\vec{\mathcal{A}} \models^* \psi_2(\vec{a})$.

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- If $\varphi := \neg\psi$, then $\vec{\mathcal{A}} \models^* \varphi(\vec{a})$ if and only if it's not the case that $\vec{\mathcal{A}} \models^* \psi(\vec{a})$.

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- If $\varphi := \exists y \psi(y, \vec{x})$, then $\vec{\mathcal{A}} \models^* \varphi(\vec{a})$ if and only if there is a $b \in \mathcal{A}_{i+1}$ and $\vec{\mathcal{A}}^{[i+1,n-1]} \models^* \psi(b, \vec{a})$.

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- If $\varphi := \forall y \psi(y, \vec{x})$, then $\vec{\mathcal{A}} \models^* \varphi(\vec{a})$ if and only if for all $j \in [i, n - dp(\psi)]$, and all $b \in \mathcal{A}_j$ we have that $\vec{\mathcal{A}} \models^* \psi(b, \vec{a})$.

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2. $\forall x, y\,(S(x) = S(y) \to x = y)$
3. $\forall x(x \neq 0 \to \exists y(x = S(y))$
4. $\forall x\,(x + 0 = x)$
5. $\forall x, y\,(x + S(y) = S(x + y))$
6. $\forall x\,(x \times 0 = 0)$
7. $\forall x, y\,(x \times S(y) = (x \times y) + x)$

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Note the depth of each of these axioms is 1 or 2.

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## Proposition

*Let $\vec{m}$ be a square increasing sequence of numbers of length at least 3 and let $\vec{\mathcal{M}}_{\vec{m}}$ be the associated square increasing model. We have that $\vec{\mathcal{M}}_{\vec{m}} \models^* \mathrm{Q}$ and $\vec{\mathcal{M}}_{\vec{m}} \models^*$ "$<$ is a linear order with no greatest element".*

# A Worked Example

Let's see how fulfillment works in practice.

## Proposition

*Let $\vec{m}$ be a square increasing sequence of numbers of length at least 3 and let $\vec{\mathcal{M}}_{\vec{m}}$ be the associated square increasing model. We have that $\vec{\mathcal{M}}_{\vec{m}} \models^* \mathrm{Q}$ and $\vec{\mathcal{M}}_{\vec{m}} \models^*$ "$<$ is a linear order with no greatest element".*

The most surprising of these is the last one. Even though $\vec{\mathcal{M}}_{\vec{m}}$ is finite, it fulfills that $<$ is infinite. Let's first show that $\vec{\mathcal{M}}_{\vec{m}}$ fulfills the first axiom of Q as a warm up and then check that it fulfills this one.

# A Worked Example

Let's see how fulfillment works in practice. For simplicity assume $m_0 > 1$ (nothing important hinges on this).

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Recall that axiom 1 is $\forall x \, 0 \neq S(x)$.

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Recall that axiom 1 is $\forall x\, 0 \neq S(x)$. This statement has only the parameter $0 \in \mathcal{M}_{m_0}$ and the term $S(x)$ so every term applied to the parameter appears in $M_{m_0}$ i.e. 0 and $1 = S(0)$ are both in $\mathcal{M}_{m_0}$ (this is where we used $m_0 > 1$ otherwise we would have to go to $\mathcal{M}_{m_1}$).

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**Lemma ($ACA_0$)**

*Let $\varphi$ be an $\mathcal{L}$-sentence. Then $\varphi$ has a model if and only if it has a $(\mathcal{L}, n)$-model for all $n > dp(\varphi)$.*

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Fix a standard proof system for first order logic formalizable in PA (say a Hilbert style system). By applying the conservativity of $ACA_0$ over PA and the arithmetized completeness theorem we get as a corollary of this lemma the following.

**Lemma (PA)**

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1. *If $\nvdash \varphi$, then for all sufficiently large $n$ there is an $(\mathcal{L}, n)$-model of $\neg\varphi$.*

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## Lemma (PA)

*Let $\varphi$ be an $\mathcal{L}$-sentence.*

1. *If $\nvdash \varphi$, then for all sufficiently large $n$ there is an $(\mathcal{L}, n)$-model of $\neg\varphi$.*
2. *If $\vdash \varphi$, then every $(\mathcal{L}, n)$-model fulfills $\varphi$ for all sufficiently large $n$.*

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As a consequence of this result we obtain the Completeness Theorem for $\models^*$.

**Theorem**

*An $\mathcal{L}$ sentence $\varphi$ is provable if and only if for all $n > dp(\varphi)$, all $(\mathcal{L}, n)$-models fulfill $\varphi$.*

As a consequence we get an important result that will be used later. Note that the significance of this theorem is that it is provable in PA.

**Corollary**

*The statement "For all finite subsets $\Gamma \subseteq \mathrm{PA}$ and all $n > \max\{dp(\gamma) + 1 \mid \gamma \in \Gamma\}$, $\Gamma$ has an $(\mathcal{L}, n)$-model" is equivalent to $\mathrm{con}(\mathrm{PA})$.*

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We will need one more result about $(\mathcal{L}, n)$-Models before we move to applications, *the finite model lemma*. This lemma is perhaps the most important as it will be used to bound the complexity of statements we wish to prove are independent. It essentially is a version of Löwenheim-Skolem for $(\mathcal{L}, n)$-models which allows for any sentence $\sigma$ one to replace a given $(\mathcal{L}, n)$-model $\vec{\mathcal{A}} \models^* \sigma$ with a new $(\mathcal{L}, n)$-model $\vec{\mathcal{B}}$ so that the cardinality of $\mathcal{B}_i$ is computable in $i$ and $\sigma$ (and a few other parameters).

In the statement on the next slide I will assume that $\vec{\mathcal{A}}$ has an external well order and use it implicitly, referring for example to "the least element of $\vec{\mathcal{A}}$ so that...holds". Note that in PA and $ACA_0$ one can assume this for free.

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**Lemma (The Finite Model Lemma)**

*Let $n$ be a natural number and $\varphi$ be an $\mathcal{L}$-sentence of depth at most $n-2$. Let $|\mathcal{L}|$ denote the cardinality of the signature of $\mathcal{L}$ and let $j$ be the largest size of an arity of a function symbol. Given any $(\mathcal{L}, n)$-model $\vec{\mathcal{A}}$, there is another $(\mathcal{L}, n)$-model $\vec{\mathcal{B}}$ so that the following hold:*

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- $\mathcal{B}_0$ *has cardinality at most* $|\mathcal{L}|$
- $\mathcal{B}_{i+1}$ *has cardinality at most* $2(\Sigma_{m=1}^{i-1}\binom{i}{m}) + (|\mathcal{B}_i| + |\mathcal{L}||\mathcal{B}_i|^j)^{|\varphi|}(1 + (2^{|\mathcal{B}_i|^{|\varphi|}}|\varphi|))$

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- *For every subformula $\psi$ of $\varphi$ $\vec{\mathcal{B}} \models^* \psi$ if and only if $\vec{\mathcal{A}} \models^* \psi$*

*Moreover, given $\varphi$, $\mathcal{L}$ and $\mathcal{A}$, the procedure for producing $\mathcal{B}$ is computable.*

# The Finite Model Lemma

The bounds are probably not best possible (and in any case the account for additional bells and whistles in the construction that I'm sweeping under the run today for the sake of presentation). What matters is that they are primitive recursive in $i$, $|\varphi|$, $|\mathcal{L}|$, $m$ and $j$. In particular they do not depend on $n$ or $\vec{\mathcal{A}}$. In what follows, I denote by $Col(i, j, k, l)$ the primitive recursive function giving these bounds where $i$ is the index of the sequence, $j$ is the greatest arity of a function symbol in $\mathcal{L}$, $k = |\varphi|$, and $l = |\mathcal{L}|$. In other words for all $i < n$ the lemma states that $|\mathcal{B}_i| < Col(i, j, |\varphi|, |\mathcal{L}|)$ (*Col* for "collapse").

# The Finite Model Lemma

## Proof Sketch.

Define the sequence $\mathcal{B}_i$ for $i < n$ by induction. It will be clear from the construction that this procedure can be carried out recursively, given knowledge of $\mathcal{A}$, $\mathcal{L}$ and $\varphi$.

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Now let $\mathcal{B}_{i+1}$ be $\mathcal{B}_i^*$ alongside all such $b$'s and $c$'s. □

# The Finite Model Lemma

Using the finite model lemma, if $\varphi$ has an $(\mathcal{L}, n)$-model $\vec{\mathcal{A}}$ which is linearly ordered by $<$, then it has one whose domain is a finite initial segment of the natural numbers via the isomorphism induced by the unique order preserving bijection between the domain of the model $\vec{\mathcal{B}}$ obtained by the computable procedure described in the finite model lemma and the initial segment is of length $|\mathcal{B}_{n-1}| < Col(n-1, |\varphi|, k, |\mathcal{L}|, n)$. Such a structure is called *the F-collapse* of $\vec{\mathcal{A}}$ for $\varphi$ ("F" for fulfillment).

The existence of an F-Collapse is what will allow us to bound the existential quantifier in a $\Pi^0_2$ sentence to get a $\Pi^0_1$ one.

Recall that *the Paris-Harrington Principle*, PH, is the statement that for all $e, k, r$ there is an $N$ so that every partition $P : [N]^e \to r$ there is a $H \subseteq N$ which is homogenous, of size at least $k$ and

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*The Paris-Harrington Theorem* is the statement that PH is independent of PA. Let me recall briefly the proof that PH is true in the standard model of PA. Note that this proves half of the Paris-Harrington Theorem since it shows PA does not prove ¬PH.

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**Theorem (Paris-Harrington, 1977)**

PH *implies con*(PA) *and so, in particular, it is not provable in* PA *and hence* PH *is independent of* PA*.*

Towards this end define the theory $\mathrm{PA}_k^{PF}$ to be the axioms of Q plus the first $k$ instances of parameter free least number principle: $LNP(\varphi) := \exists x \varphi(x) \to \exists x \forall y(\varphi(x) \wedge (\varphi(y) \to x \leq y))$ where $\varphi$ is one of the first $k$ formulae relative to some primitive recursive ordering of the formulas of $\mathcal{L}$. It's well known that PA is equivalent to $\bigcup\{\mathrm{PA}_k^{PF} \mid k \in \omega\}$.

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**Theorem**

PA + PH *implies that for each* $k$ *and all sufficiently large* $n$ *there is a* $(\mathcal{L}, n)$*-model of* $\mathrm{PA}_k^{PF}$*.*

# A Proof of the Paris-Harrington Theorem

Before beginning on the proof of this theorem, let us isolate a slightly modified version of PH which we will need. In the original paper by Paris-Harrington they show that PH already implies (over base theory PA) several seemingly stronger statements. The following one is a special case of their general theorem.

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Given a number $N$ let is denote by $[N]^e_{sqInc}$ the set of $e$-sized subsets of $N$ which, when placed in ascending order, are square increasing.

**Lemma**

PH *implies that for every* $e, k, r, m$ *there is an* $N$ *so that every function* $F : [N]^e_{sqInc} \to r$ *there is a* $H \subseteq N$ *so that* $H$ *is square increasing,* $F \restriction [H]^e$ *is constant,* $H$ *contains only elements larger than* $m$ *and so that the cardinality of* $H$ *is larger than* $\min(H) + k$*.*

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• Since we know that square increasing models fulfill Q it's therefore enough to show that for each finite $\Gamma$ of $\mathcal{L}$ sentences, and each sufficiently large $n$ there is a square increasing $(\mathcal{L}, n)$-model of $\Gamma$.

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• Since we know that square increasing models fulfill Q it's therefore enough to show that for each finite $\Gamma$ of $\mathcal{L}$ sentences, and each sufficiently large $n$ there is a square increasing $(\mathcal{L}, n)$-model of $\Gamma$.

• For simplicity let us fix a sentence $\varphi$ and show this for just this $\varphi$. The modification from one formula to finitely many is straightforward.

• Fix $n$ larger than $dp(\varphi) + 3$. If there is a square increasing $(\mathcal{L}, n)$-model $\vec{\mathcal{M}}_{\vec{m}} \models^* \neg\exists x\, \varphi(x)$ then we're done so suppose that all such models fulfill $\exists x \varphi(x)$. We need to show that they fulfill that there is a least such $x$. Also, fix a number $m$ large enough that all terms in $\varphi$ are definable in $\mathcal{M}_m$.

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• For any square increasing sequence $\vec{m} = m_0 < ... < m_{n-2}$, with $m_0 > m^2$ let us define $F_\varphi(\vec{m}) = \min\{x < m_0 \mid \vec{\mathcal{M}}_{\vec{m}} \models^* \varphi(x)\}$. By the assumption $F_\varphi$ is defined on all square increasing sequences of length $n-1$ with first element at least $m^2 + 1$.

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• Now for such a square increasing sequence $m_0 < ... < m_{n-2} < m_{n-1}$ of length $n$ let $F'_\varphi(m_0, m_1, ..., m_{n-2}, m_{n-1}) =$

$$\begin{cases} 0, & F_\varphi(m_0, m_2, m_3, ..., m_{n-1}) = F_\varphi(m_0, m_1, m_3, ..., m_{n-1}) \\ 1, & \textit{otherwise} \end{cases}$$

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• This is a two coloring of $n$-tuples.

# A Proof of the Paris-Harrington Theorem

Applying PH let $N$ be such that all $F'_\varphi$ restricted to $[N]^n_{SqInc}$ has a homogenous subset $H \subseteq N$ so that every element is larger than $m^2$, and whose cardinality is larger than $\min(H) + n + 5$.

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**Claim**

*$F'_\varphi \upharpoonright [H]^n$ is identically 0.*

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## Proof.

Otherwise it's 1. But that means that, if $H = \{m_0 < m_1 < ... < m_k\}$ we have that the set $\{F_\varphi(m_0, m_l, m_{k-n-2}..., m_k) \mid 1 \leq l \leq k - n - 3\}$ is a subset of $m_0$ of size $k - n - 4$. But by construction $k > m_0 + n + 5$ so this contradicts the pigeonhole principle. □

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• Let $m_0 < m_1 < ... < m_n \in H$ and let $\vec{m}$ be the associated square increasing sequence. I claim that $\vec{\mathcal{M}}_{\vec{m}}$ fulfills that $x_0 := F_\varphi(m_1, ..., m_n)$ is the minimal $x$ so that $\varphi(x)$.

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• But that is a contradiction to the fact that $F'_\varphi \restriction [H]^n$ is identically 0 since in this case we actually have that $y < x_0$ (in $\mathcal{M}$) and by the previous claim, $x_0$ was the least so that any $n$-tuple of elements from $H$ fulfilled $\varphi(x_0)$.

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• To conclude the proof observe that it follows that $\langle M_m, \vec{\mathcal{M}}_{\vec{m}} \rangle \models^* LNP(\varphi)$.

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How can we do this?

Color $(\mathcal{L}, n)$-models and use the finite model lemma!

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Let $r$, $n$, and $N$ be natural numbers and let $\varphi(x)$ an $\mathcal{L}$-formula.

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A *bounded* $(n, \varphi)$*-coloring in* $r$ colors *on* $N$ is a function $F$, so that the following conditions hold:

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2. The range of $F$ is $r$.
3. Boundedness: For each $k \geq n$ and every $(\mathcal{L}, k)$-model $\vec{\mathcal{A}} = \langle \mathcal{A}_0, ..., \mathcal{A}_{k-1} \rangle$, so that all of the sub $n$-tuples of $\vec{\mathcal{A}}$ are in the domain of $F$, we have that if $\vec{\mathcal{B}} = \langle \mathcal{B}_0, ..., \mathcal{B}_{k-1} \rangle$ is the F-collapse of $\vec{\mathcal{A}}$ for $\exists x \varphi(x)$ then for any $i_0 < i_1 < ... < i_{n-1} < k$ we have that $F(\mathcal{A}_{l_{i_0}}, ..., \mathcal{A}_{l_{i_{n-1}}}) = F(\mathcal{B}_{l_{i_0}}, ..., \mathcal{B}_{l_{i_{n-1}}})$.

# A True, Unprovable $\Pi^0_1$-Sentence

Observe that to say that "$F$ is a bounded $(n, \varphi)$-coloring in $r$ colors on $N$" is $\Delta_0$ with parameters $n, \varphi, r$ and $N$ since every quantifier in the definition can be bounded by $2^{N^2}$.

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**Definition**

The *Bounded Coloring Principle*, denoted BCP is the statement that for all $r, n, \mathcal{L}, \varphi, j, m, k$ if $k \geq n, |\mathcal{L}| + m$, the largest arity of a function symbol in $\mathcal{L}$ is $j$ and $F$ is a $(n, \varphi)$-bounded coloring in $r$ colors on $Col(k, j, |\exists x\, \varphi(x)|, |\mathcal{L}|, n) + 1$

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For a fixed $r, n, \mathcal{L}, \varphi, j, m, k$ and $N$ let us denote the conclusion of BCP by BCP$(r, n, \mathcal{L}, \varphi, j, k, m, N)$.

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For a fixed $r, n, \mathcal{L}, \varphi, j, m, k$ and $N$ let us denote the conclusion of BCP by $\mathrm{BCP}(r, n, \mathcal{L}, \varphi, j, k, m, N)$. Note that BCP is the statment

$$\forall r, n, \mathcal{L}, \varphi, j, m, k\, \mathrm{BCP}(r, n, \mathcal{L}, \varphi, j, k, m, Col(k, j, |\exists x\, \varphi(x)|, |\mathcal{L}|, n) + 1))$$

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and so in particular it's $\Pi^0_1$. We will sketch a proof of the following theorem.

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**Theorem (S.)**

*The statement* BCP *is true in the standard model but* $\mathrm{PA} + \mathrm{BCP}$ *implies* $\mathrm{con}(\mathrm{PA})$*. In particular,* BCP *is independent of* PA*.*

# A True, Unprovable $\Pi^0_1$-Sentence

The main tool in proving the above theorem is a reduction of BCP to a $\Pi^0_2$ sentence with the primitive recursive bound given by the collapse function removed. The point is that the definition of boundedness alongside the technology of the finite model lemma is tailored for this.

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The main tool in proving the above theorem is a reduction of BCP to a $\Pi^0_2$ sentence with the primitive recursive bound given by the collapse function removed. The point is that the definition of boundedness alongside the technology of the finite model lemma is tailored for this.

**Lemma**

*The principle* BCP *is equivalent to the statement, which I call* BCP$'$, *that* $\forall r, n, \mathcal{L}, \varphi, j, m, k \exists N \, \mathrm{BCP}(r, n, \mathcal{L}, \varphi, j, k, m, N)$

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**Proof.**

Clearly BCP implies BCP$'$. For the converse, suppose BCP$'$ holds, fix $r, n, \mathcal{L}, j, k, m$ and let $N$ be large enough to witness BCP$'$.

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Clearly BCP implies BCP′. For the converse, suppose BCP′ holds, fix $r, n, \mathcal{L}, j, k, m$ and let $N$ be large enough to witness BCP′. We need to show that already there is a homogeneous sequence of structures all of whose universes are contained in $Col(k, j, |\exists x\, \varphi(x)|, |\mathcal{L}|, n) + 1$.

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# A True, Unprovable $\Pi^0_1$-Sentence

The rest of the proof follows the along the same lines as the proof of the Paris-Harrington Theorem, replacing PH with BCP′. That BCP′ is true in the standard model is a similar tree argument so I won't repeat it. Let's focus on the "unprovable" part.

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As before we will show that for any $\varphi$ and any sufficiently large $n$ we can find an $(\mathcal{L}, n)$-model of $\bigwedge \mathsf{Q} \wedge LNP(\varphi)$. Specifically we will show the following.

# A True, Unprovable $\Pi^0_1$-Sentence

The rest of the proof follows the along the same lines as the proof of the Paris-Harrington Theorem, replacing PH with BCP′. That BCP′ is true in the standard model is a similar tree argument so I won't repeat it. Let's focus on the "unprovable" part.

As before we will show that for any $\varphi$ and any sufficiently large $n$ we can find an $(\mathcal{L}, n)$-model of $\bigwedge \mathrm{Q} \wedge LNP(\varphi)$. Specifically we will show the following.

**Lemma**

*The statement* BCP′ *implies that that for all sufficiently large n there is an* $(\mathcal{L}, n)$ *model of* $\bigwedge Q \wedge LNP(\varphi)$.

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To prove this lemma, let us assume BCP′. If $\mathsf{Q} \cup \{\neg\exists x\, \varphi(x)\}$ is consistent we are done hence let us suppose that $\mathsf{Q} \vdash \exists x \varphi(x)$.

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- Let $n$ be larger than the depth of $\bigwedge \mathsf{Q} \wedge \exists x\, \varphi(x)$.

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• Let $n$ be larger than the depth of $\bigwedge \mathrm{Q} \wedge \exists x\, \varphi(x)$. Fix an $m$ large enough so that all terms in $\varphi$ are defined in $\mathcal{M}_m$. Enlarge $\mathcal{L}$ with a constant symbol for each number $a < m^2$. This ensures that any $F$-collapse of an $(\mathcal{L}, n)$-model for $\exists x\, \varphi(x)$ will have all relevant parameters in the first model.

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- For an $(\mathcal{L}, n)$-model $\vec{\mathcal{A}} = \langle \mathcal{A}_0, ..., \mathcal{A}_{n-1} \rangle \models^* Q$ with $\vec{\mathcal{M}}_m \subseteq \mathcal{A}_0$ let $F(\vec{\mathcal{A}})$ be the least $b \in \mathcal{A}_0$ with respect to the linear ordering $<$ as defined on the top model of $\vec{\mathcal{A}}$ so that $\vec{\mathcal{A}} \models^* \varphi(b)$.

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Now define a bounded $(n+1, \bigwedge Q \wedge \varphi)$-coloring $F$ on some sufficiently large $N$ (large enough to run the following argument as given by BCP′) as follows:

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Now define a bounded $(n+1, \bigwedge Q \wedge \varphi)$-coloring $F$ on some sufficiently large $N$ (large enough to run the following argument as given by BCP$'$) as follows:

$F'(\langle \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \rangle) =$

$$\begin{cases} 0, & F(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_3, ..., \mathcal{A}_n) = F(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_3, ..., \mathcal{A}_n) \\ 1, & otherwise \end{cases}$$

Now, by BCP$'$ we can choose $N$ large enough so that $F$ has a homogeneous sequence $\vec{H} = \langle \mathcal{A}_0, ..., \mathcal{A}_{k-1} \rangle$ of length $k > |\mathcal{A}_0| + m + n + 1$.

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Since $|\mathcal{A}_0| < k - m - n - 1$ we must have that $F \upharpoonright [H]^{n+1} \equiv 0$ by a pigeonhole argument similar to the one for PH.

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Since $|\mathcal{A}_0| < k - m - n - 1$ we must have that $F \upharpoonright [H]^{n+1} \equiv 0$ by a pigeonhole argument similar to the one for PH.

However then we can run an analogous argument to one given for PH to show that the first $n$-tuple of elements from $\vec{H}$, coupled with $M_m$ as the minimal element of the sequence will satisfy $LNP(\varphi)$, thus completing the proof of the lemma and hence the sketch of the proof of the theorem.

# In Closing...

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# In Closing...

Using a similar framework one can prove the independence from PA of many of the known Paris-Harrington like statements.

• In particular the independence of the Regressive Ramsey Theorem of Kanamori and McAloon can be shown in this way and, a $\Pi^0_1$, $(\mathcal{L}, n)$-model modification can be given for it which is also independent.

Many other $\Pi^0_1$ examples of this type seem in reach due to the finite model lemma.

# Thank You for Your Attention!

# References

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