Specializing Wide Trees Without Adding Reals

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Introduction

The main question I want to discuss in this talk is "When can a wide Aronszajn tree be specialized by a forcing which does not add reals?"

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- Given cardinals κ and λ, a $(\kappa, \leq\lambda)$-*tree* T is a tree of height κ with levels of size $\leq\lambda$. I also sometimes write (κ, λ)-tree or $(\kappa, < \lambda)$-tree with the obvious meanings.
- I say that T is a $(\kappa, \leq\lambda)$-*Aronszajn tree* if it has no branch of length κ.
- A tree is called a κ-tree if it is a $(\kappa, < \kappa)$-tree.
- In slight departure from normal notation, I will call an $(\omega_1, \leq \lambda)$-Aronszajn tree T an *Aronszajn* tree (for any width λ). If all the levels of T are countable, I will call T a *thin* Aronszajn tree. If there are uncountable levels then I call T *wide*.

Souslin and Special Trees

Fix an Aronszajn tree T (so, height ω_1 but no assumed width restrictions).

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- We say that T is *Souslin* if every antichain in T is countable (i.e. T has the ccc).
- We say that T is *special* if any of the following equivalent conditions hold:

1. T can be decomposed into countably many antichains.
2. There is a function $f : T \to \omega$ which is injective on chains.
3. There is a function $f : T \to \mathbb{Q}$ which is monotone increasing on chains. Such an f as in the second or third condition is called a *specializing function*.

Souslin and Special Trees

- Note that a special tree is never Souslin since at least one of the countably many antichains covering T must be uncountable.
- Also, if T is special then forcing with T collapses ω_1 since given any cofinal branch $b \subseteq T$ in the extension, and any specializing function $f : T \to \omega$, the restriction $f \upharpoonright b : b \to \omega$ would be an injection of a set of size $(\omega_1)^V$ into ω. So treating T as a forcing notion it's not even proper.
- Since Souslin trees are ccc this is another difference.

Special Aronszajn Trees Exist

• Special thin Aronszajn trees exist in ZFC, this is Aronszajn's original construction.

• Roughly the idea is to consider the tree consisting of bounded well-ordered sequences of rationals with a maximal element ordered by end extension. Clearly this has no branch of length ω_1 but it does have elements of arbitrary height. However, the levels of this tree are size continuum so we use the countability of $\mathbb{Q}$ to "thin out" the tree.

• For each $q \in \mathbb{Q}$, inductively at each limit stage pick one sequence bounded by q to extend (adding q as the maximal element). It's straightforward to check that the function sending each such sequence to its max is a specializing function.

The Baumgartner-Malitz-Reinhardt Poset

• Any ZFC construction of an Aronszajn tree of width $<2^{\aleph_0}$ (so in particular a thin tree) must be special since it is consistent that *all* Aronszajn trees of width $<2^{\aleph_0}$ are special (Baumgartner-Malitz-Reinhardt).

• For any Aronszajn tree T, let $\mathbb{S}_T$ be the forcing whose conditions are finite partial functions $f : T \to \mathbb{Q}$ so that if $t, s \in \mathrm{dom}(f)$ and $t <_T s$ then $f(t) < f(s)$ (in the rational ordering).

• Baumgartner-Malitz-Reinhardt showed that $\mathbb{S}_T$ is ccc for any T so $\mathsf{MA} + \neg\mathsf{CH}$ implies that all Aronszajn trees of width $<2^{\aleph_0}$ are special.

• The posets of the form $\mathbb{S}_T$ have found uncountably many applications since their introduction. In particular they are used in key steps in the proofs that PFA implies there are no Kurepa trees, ω_2 has the tree property, $\square^*_{\omega_1}$ fails and more.

The Baumgartner-Malitz-Reinhardt Poset

- However,this forcing adds $|T|$-many reals.
- In fact it's not hard to see that, given any countable set $B \subseteq T$ enumerated $B = \{b_n \mid n < \omega\}$, the function in the extension $n \mapsto g(b_n) \bmod 2$, where $g : T \to \omega$ is the generic specializing function, will be a Cohen real.
- It's therefore interesting to ask whether there is a similar poset for specializing trees that does not add reals, so that these results could be studied in the CH context.

Another reason to ask about such a poset is its connection to the Souslin Problem.

Killing Souslin Trees

Recall the following:

Theorem

1. *(Jensen) Under* $\diamondsuit$ *there is a Souslin tree.*
2. *(Solovay-Tennenbaum) Under* $\mathrm{MA}_{\aleph_1}$ *there are no Souslin trees.*

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Observe that under $\diamondsuit$, CH holds whereas under $\mathrm{MA}_{\aleph_1}$ CH fails. Thus was not clear whether the existence Souslin trees follows from CH. Indeed, it was not originally clear whether or not $\diamondsuit$ and CH were equivalent.

Jensen's Model

Jensen showed that CH does not imply there are Souslin trees (and hence it does not imply $\diamondsuit$). His proof was one of the first applications of countable support iterations and special trees.

Theorem (Jensen)

Assume $V = L$. There is a countable support iteration $\mathbb{P}$ so that $\mathbb{P}$ adds no reals and $\Vdash_{\mathbb{P}}$ "All thin Aronszajn trees are special".

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The proof of this theorem is rather complicated (i.e. I frankly don't understand it, maybe it's actually easy). In trying to work through it Shelah developed a more streamlined approach, removing the $V = L$ assumption and isolating a class of proper posets which don't add reals and can be iterated with countable support: **the dee-complete and $<\omega_1$-proper posets**.

Shelah's Model

The definition of dee-complete and $<\omega_1$-proper is rather involved and I won't give it in this talk. However, to understand the theorem below (and the rest of the talk) it suffices to know that it is a class of proper posets which can be iterated without adding reals and includes all σ-closed posets.

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Theorem (Shelah)

1. For any thin Aronszajn tree T there is a dee-complete and $<\omega_1$-proper poset $\mathbb{P}_T$ which specializes T.
Thus,
2. Under the forcing axiom for this class, DCFA *(the dee-complete forcing axiom) all thin Aronszajn trees are special.*
Moreover,
3. Since the class is iterable without adding reals, DCFA *is consistent with* CH.

So that Solves It?

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- In fact they do not.
- The ccc poset works irrespctive of the widths of the tree being specialized whereas the countability of the levels is essential in the posets used by Jensen and Shelah.
- Moreover, in all three applications of PFA mentioned above, the forcing needed was a version of the ccc poset for specializing *wide* trees.

A Non-Special Wide Tree

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Theorem (Todorčević (?))

There is (in ZFC) an Aronszajn tree T of width continuum that cannot be specialized without adding reals.

A Non-Special Wide Tree

Proof (Sketch).

• Fix a stationary/co-stationary subset $S \subseteq \omega_1$ and let $T = T(S)$ be the set of closed bounded countable sequences $c \subseteq S$ ordered by end extension.

•This is a tree of height ω_1 and it has no cofinal branch since the union of the sequences on such a branch would be a club contained in S (but $\omega_1 \setminus S$ is stationary).

• Forcing with this tree is equivalent to the standard forcing to shoot a club through S and this forcing does not collapse ω_1. Therefore T is not special.

• The elements of T are coded by reals and the order and definition of T are absolute, hence in any forcing extention with the same reals T is unchanged so it must still not be special.

□

Questions, Questions, Questions

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So we're left with the following:

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2. Can such a forcing be used to show that a forcing axiom compatible with CH has similar consequences to those of PFA that use the ccc forcing?

In the time left I would like to prove a partial solution to these two questions.

The Main Theorems

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Corollary

DCFA *implies there are no Kurepa trees.*

This latter result was proved by Shelah by a different method and under the additional assumptions that $2^{\aleph_1} = \aleph_2$ and CH hold.

The Proof: Partial Specializing Functions

Here is a sketch of the proof of the theorem. The poset constructed in the proof is very similar to (and inspired by) one used by Abraham and Shelah (itself a variation on Shelah's original poset). The only difference is the accomodation made to allow a wider tree.

The Proof: Partial Specializing Functions

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- Fix trees of height ω_1, $S \subseteq T$ so that T is thin, potentially with cofinal branches, and S is a (wide) Aronszajn tree.
- The intuition is that we want to force with partial specializing functions $f : S \to \mathbb{Q}$ but use the structure of T to guide the construction. More formally:

Definition

A *partial specializing function* of height α is a function $f : T_{\leq \alpha} \cap S \to \mathbb{Q}$ which is strictly increasing on linearly ordered chains. We write $ht(f)$ to denote the height of f.

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- Forcing with these functions alone won't work.
- The issue is roughly as follows. Suppose $\langle \alpha_n \mid n < \omega \rangle$ is an increasing sequence of ordinals with supremum α and let $t \in T_\alpha \cap S$ be a node on the α^{th}-level of T so that for all $n < \omega$ $t \upharpoonright \alpha_n \in S$. If f_n are partial specializing functions of height α_n respectively, so that $f_n \subseteq f_{n+1}$ then by density we can ensure that $\langle f_n(t \upharpoonright \alpha_n) \mid n < \omega \rangle$ diverges to ∞ but then there is no way to assign a value to t to continue specializing the tree.

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1. A (possibly partial) function $h : T_\beta \to \mathbb{Q}$ *projects into* S if for each $t \in \mathrm{dom}(h)$ there is an $\alpha < \beta$ so that $t \upharpoonright \alpha \in S$. We say that such an h *bounds* a partial specializing function if $\beta \geq \alpha + 1$ and for all t in the domain of h whose projection to the $\alpha + 1^{\mathrm{st}}$ level is in S we have that $h(t) > f(t \upharpoonright \alpha + 1)$.

2. A *requirement* H of height β and arity $n = n(H) \in \omega$ is a countably infinite family of finite functions $h : T_\beta \to \mathbb{Q}$ which project into S and whose domains have size n.

The Proof: Defining the Poset

Definition

3. A partial specializing function f *fulfills* a requirement H if the height of f is at most the height of H and for every finite $\tau \subseteq T_\beta$, β the height of H, there is an $h \in H$ bounding f whose domain is disjoint from τ.

4. A *promise* is a function Γ defined on a tail set of countable ordinals, the first of which we denote $\beta = \beta(\Gamma)$ so that for each $\gamma \geq \beta$, $\Gamma(\gamma)$ is a countable set of requirements of height γ and if $\gamma' \geq \gamma$ then $\Gamma(\gamma) = \Gamma(\gamma') \upharpoonright \gamma$ i.e. every $H \in \Gamma(\gamma)$ there is some $H' \in \Gamma(\gamma')$ so that $H' = \{h \upharpoonright \gamma' \mid h \in H\}$ where each such projection is defined.

5. A partial specializing function f *keeps* a promise Γ if $\beta(\Gamma) \geq ht(f)$ and f fulfills every $H \in \Gamma(\gamma)$ for all $\gamma \geq \beta$. Note that by the projection property given in the definition of a promise, to keep a promise it suffices to fulfill the requirements at the first level.

The Proof: Defining the Poset

Finally I can define the poset.

Definition

The forcing $\mathbb{P} = \mathbb{P}_{T,S}$ consists of pairs $p = (f_p, \Gamma_p)$ where f_p is a partial specializing function, Γ_p is a promise and f_p keeps Γ. We write $ht(p)$ for $ht(f_p)$. The order is as follows: we let $p \leq q$ if $f_p \supseteq f_q$ and for all $\gamma \geq ht(q)$ $\Gamma_p(\gamma) \supseteq \Gamma_q(\gamma)$.

The Proof: Some Lemmas

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The first of these is easily accomplished. A relatively straightforward density argument shows that:

Lemma (The Extension Lemma)

Suppose $p \in \mathbb{P}$ of height α and let $\beta \geq \alpha$. Then there is a $q \leq p$ of height β. Moreover, if $g : T_\beta \to \mathbb{Q}$ is a finite function bounding f_p then q can be found so that h bounds f_q as well.

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- To see that $\mathbb{P}$ specializes S the first part of the extension lemma suffices (though we haven't showed that $\mathbb{P}$ is proper yet so this might be trivial since ω_1 could be collapsed).
- However, the "moreover" part is important: it tells us that it's not dense to make the generic specializing function grow faster than some predecided bound when extending a condition to a higher level.

The Proof: Some Lemmas

Souping up this lemma gives the following, which is key.

Lemma (The Submodel Lemma)

Let θ be sufficiently large and let $M \prec H_\theta$ be countable containing $T, S, \mathbb{P}, etc.$ Let $p \in \mathbb{P} \cap M$ and let $\delta = M \cap \omega_1$. Note that $M \cap T = T_{<\delta}$. Let $D \in M$ be a dense open subset of $\mathbb{P}$ and let $h : T_\delta \to \mathbb{Q}$ be a finite function bounding f_p. Then there is an extension $q \in D \cap M$ so that f_q is also bounded by h.

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Intuitively, this lemma says that for $M \prec H_\theta$ as in the hypothesis, one can always extend any condition $p \in M \cap \mathbb{P}$ into any given $D \in M$ which is dense open without violating some fixed bound h on the $(\omega_1)^M$th level.

The Proof: Properness and not adding Reals

Proving $\mathbb{P}$ is proper and does not add reals follows by bootstrapping this lemma and utilizing the countability of the levels of T.

- The key point is that since T has countable levels, $T \cap M = T_{<\delta}$ and since S embeds into T, in order to bound $S \cap M$ we only need to ensure there is a bound on T_δ. The details are as follows.

Lemma

$\mathbb{P}$ is proper and adds no new reals. More precisely, if θ be sufficiently large, $M \prec H_\theta$ is countable containing $T, S, \mathbb{P}$, etc and $p \in \mathbb{P} \cap M$ then there is a $G \subseteq \mathbb{P} \cap M$ generic with $p \in G$ and a $q \in \mathbb{P}$ so that for all $r \in G$ $q \leq r$.

The Proof: Properness and Not Adding Reals

Proof (Sketch).

- Fix θ, M, p etc as in the statement of the lemma and let $\delta = \omega_1 \cap M$. By the remark preceding the statement of the lemma it suffices to simultaneously define a total function h on T_δ and a generic $G \ni p$ so that that $\bigcup G$, which is a specializing function on $T_{<\delta}$ is bounded by h. Once this is done, one can use the bound to extend $\bigcup G$ to a condition of height δ, which is the requisite q.
- To do this, let $\langle D_n \mid n < \omega \rangle$ enumerate the dense open subsets of $\mathbb{P}$ in M and inductively define finite functions h_n and a decreasing sequence of conditions $p_n \leq p$ so that $p_n \in D_n$ and is bounded by h_n. The inductive stage of this construction is exactly the submodel lemma.

□

Two Easy Corollaries

This theorem has a few easy corollaries. First, is the following, odd ZFC result which may be of independent interest. I have no idea if this was previously known.

Corollary

The tree $T(S)$ from the example due to Todorčević does not embed into an ω_1-tree.

Two Easy Corollaries

The second strengthens the statement from the Jensen/Shelah models.

- I won't prove it, but as mentioned $\mathbb{P}$ from the theorem has a stronger property than being just proper and not adding reals, it is dee-complete and $<\omega_1$-proper.
- This suffices to ensure we can iterate such forcings without adding reals. As a result it follows that:

Corollary

It's consistent with CH *that all Aronszajn trees which embed into an ω_1-tree (possibly non-Aronszajn) are special.*

Kurepa Trees

Let's see a concrete example of such a pair of trees $S \subseteq T$ as well as an application of the poset $\mathbb{P}_{T,S}$.

Kurepa Trees

Let's see a concrete example of such a pair of trees $S \subseteq T$ as well as an application of the poset $\mathbb{P}_{T,S}$. Recall that a Kurepa tree is an ω_1-tree with at least $\aleph_2$-many cofinal branches. The axiom DCFA is the forcing axiom for the class of dee-complete and $<\omega_1$-proper forcing notions.

Corollary

Assume DCFA. *Then there are no Kurepa trees.*

Kurepa Trees

The proof of the corollary is almost verbatim the same as the proof that there are no Kurepa trees under PFA (due to Baumgartner), substituting in the poset $\mathbb{P}_{T,S}$ for the ccc specializing poset. As mentioned before, Shelah showed DCFA implies there are no Kurepa trees by a different proof and under additional cardinal arithmetic assumptions that hold in the natural model of DCFA.

Proof (Sketch).

- Assume DCFA holds and that T is a Kurepa tree with $\theta \geq \aleph_2$ many cofinal branches.
- Let $\mathrm{Col}(\theta, \aleph_1)$ be the standard σ-closed forcing to collapse θ to $\aleph_1$. Since this forcing notion is σ-closed it's dee-complete, $<\omega_1$-proper and adds no new branches to T hence in the extension T has $\aleph_1$-many cofinal branches.

□

Proof (Sketch), Continued.

- Baumgartner showed that any ω_1-tree with only $\aleph_1$-many cofinal branches embeds a wide Aronszajn tree, S. Moreover, if S is special then one can, from a specializing function for S, define a function $f : T \to \mathbb{Q}$ so that if $t \leq_T s, r$ and $f(s) = f(t) = f(r)$ then s and r are compatible.
- So specialize S with $\mathbb{P}_{T,S}$. We have that $\mathrm{Col}(\theta, \aleph_1) * \dot{\mathbb{P}}_{T,S}$ is dee-complete and $<\omega_1$-proper and adds a function $f : T \to \mathbb{Q}$ as in the previous bullet point. Applying DCFA we can "pull-back" to V to find such a function $f : T \to \mathbb{Q}$ (in V).

□

Kurepa Trees

Proof (Sketch), Continued.

- But now, for each cofinal branch $b \subseteq T$ let t_b be the least node so that there is a $q \in \mathbb{Q}$ such that $f(t_b) = q$ and there are $\aleph_1$-many $s \in b$ with $f(s) = q$ (such a node must exist since $|b| > |\mathrm{range}(f)|$).
- By the properties of f, if $b \neq b'$ then $t_b \neq t_{b'}$ so the mapping $b \mapsto t_b$ is an injection. But $|T| = \aleph_1$ so it must have at most $\aleph_1$-branches, contradicting the fact that T was supposed to be Kurepa.

□

Thank You and Stay Safe!

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