Strong guessing models

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Background and motivation

This is joint work with my PhD student R. Mohammadmour.

**General form of forcing axioms.** Let $\mathcal{K}$ be a class of forcing notions and $\kappa$ an uncountable cardinal.

\[
\text{FA}_{\kappa}(\mathcal{K})
\]

For every $\mathcal{P} \in \mathcal{K}$ and a family $\mathcal{D}$ of $\kappa$ dense subsets of $\mathcal{P}$ there is a filter $G$ in $\mathcal{P}$ such that $G \cap D \neq \emptyset$, for all $D \in \mathcal{D}$.

- MA$_{\kappa} \equiv$ FA$_{\kappa}(\text{ccc})$
- PFA $\equiv$ FA$_{\aleph_1}(\text{proper})$
- MM $\equiv$ FA$_{\aleph_1}(\text{stationary preserving})$

**Remark**

$\mathcal{K}$ cannot be the class of all posets or even all posets preserving $\aleph_1$. 
**PFA implies**

- $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$
- Singular Cardinal Hypothesis
- The tree property at $\aleph_2$
- the failure of $\Box(\kappa)$, for regular $\kappa > \aleph_1$.

**MM implies**

- $\text{NS}_{\omega_1}$ is $\aleph_2$-saturated
- Chang’s conjecture $(\aleph_2, \aleph_1) \rightarrow (\aleph_1, \aleph_0)$

We are looking for higher forcing axioms that have similar structural consequences. In particular we want to have $2^{\aleph_0} > \aleph_2$. 
Guessing models

We first search for some principles that follow from PFA, imply most of its structural properties, but are consistent with $2^{\aleph_0}$ being bigger than $\omega_2$. The key notion is that of a **guessing model**.

**Definition (Viale)**

Let $R$ be a model of a fragment of set theory and $M \prec R$. Let $\gamma$ be a cardinal. Let $Z \in M$ and $f : Z \to 2$ be a function.

1. $f$ is **$\gamma$-approximated** in $M$ if $f \upharpoonright C \in M$, for all $C \in \mathcal{P}_\gamma(Z) \cap M$.
2. $f$ is **guessed** in $M$ if there is $\bar{f} \in M$ such that $f \upharpoonright M = \bar{f} \upharpoonright M$.

We say that $M$ is a **$\gamma$-guessing model** if every $f \in R$ which is $\gamma$-approximated in $M$ is guessed in $M$.

**Remark**

$M \prec H_\theta$ is a $\gamma$-guessing model iff the transitive collapse $\bar{M}$ of $M$ has the $\gamma$-approximation property in the sense of Hamkins.
Write $\mathcal{P}_\kappa^*(R)$ for the set of all $M < R$ such that $M \cap \kappa \in \kappa$. For $\gamma \leq \kappa$ we let

$$\mathcal{G}_{\kappa,\gamma}(R) = \{M \in \mathcal{P}_\kappa^*(R) : M \text{ is } \gamma\text{-guessing}\}.$$ 

**Definition (Viale)**

$\text{GM}(\kappa, \gamma)$ is the statement that $\mathcal{G}_{\kappa,\gamma}(H_\theta)$ is stationary, for all sufficiently large $\theta$.

We are primarily interested in $\gamma = \omega_1$ and $\kappa = \omega_2$, i.e. $\omega_1$-guessing models of size $\omega_1$. 
Lemma (Viale)

1. If $M$ is $\aleph_0$-guessing then $\kappa_M = M \cap \kappa$ and $\kappa$ are inaccessible.
2. $M < V_\delta$ is $\aleph_0$-guessing iff $\bar{M} = V_{\bar{\delta}}$, for some $\bar{\delta}$, where $\bar{M}$ is the transitive collapse of $M$.

The following is a reformulation of Magidor’s characterization of supercompactness in terms of $\aleph_0$-guessing models.

Theorem (Magidor)

$\kappa$ is supercompact iff $GM(\kappa, \aleph_0)$ holds.

Remark

For this reason we use the term **Magidor models** for $\aleph_0$-guessing models.
### Theorem (Viale, Weiss)

PFA implies $\text{GM}(\omega_2, \omega_1)$.

### Theorem (Weiss)

$\text{GM}(\omega_2, \omega_1)$ implies

1. the failure of $\square(\lambda)$, for all regular $\lambda \geq \omega_2$.
2. $\text{TP}(\omega_2)$, in fact, $\text{TP}(\omega_2, \lambda)$, for $\lambda \geq \omega_2$.

### Theorem (Viale, Krueger)

$\text{GM}(\omega_2, \omega_1)$ implies $\text{SCH}$.

### Theorem (Cox, Krueger)

$\text{GM}(\omega_2, \omega_1)$ is consistent with $2^{\aleph_0}$ arbitrarily large.
Definition

Let $\theta > \omega_1$ be a regular cardinal. Let $N < H_\theta$ be of size $\aleph_1$.

- We say that $N$ is **internally unbounded (I.U.)** if there is an $\in$-chain of countable models $(N_\xi : \xi < \omega_1)$ such that $N = \bigcup \xi N_\xi$.
- We say that $N$ is **internally club (I.C.)** if the above sequence can be taken to be continuous.

Definition

Let $M < H_\theta$ be of size $\aleph_1$. We say that $M$ is **locally internally unbounded** if $\mathcal{P}_{\omega_1}(X) \cap M$ is cofinal in $\mathcal{P}_{\omega_1}(X \cap M)$, for every $X \in M$.

Fact

Suppose $\theta_0 < \theta_1$ are regular and $M < H_{\theta_1}$ is locally internally unbounded with $\theta_0 \in M$. Then $M \cap H_{\theta_0}$ is internally unbounded.
Theorem (Krueger)

If $M < H_\theta$ is an $\omega_1$-guessing model of size $\aleph_1$, then $M$ is locally internally unbounded.

Proof.

Let $X \in M$ and $x \in \mathcal{P}_{\omega_1}(X \cap M)$. We need to find a countable $y \in M$ with $x \subseteq y$.

Let $f : \omega \to x$ be a bijection, and set $x_n = f^{[n]}n$, hence $x_n \subseteq x_{n+1}$.

Let $A = \{x_n : n \in \omega\}$. Then $A \subseteq [X]<\omega \in M$.

- If $A$ is countably approximated in $M$, since $M$ is an $\omega_1$-guessing model, $A \in M$, and hence $x = \bigcup A \in M$. Set $y = x$.

- Otherwise there is a countable $Y \subseteq [X]<\omega$ in $M$ such that $A \cap Y \notin M$, but then $A \cap Y$ is infinite, and $x = \bigcup(A \cap Y) \subseteq \bigcup Y \in M$. Set $y = \bigcup Y$. 

\[
\square
\]
Definition (Viale)

Let \( \lambda \) be singular of cofinality \( \omega \). \( \mathcal{A} = (A(n, \alpha) : n < \omega, \alpha < \lambda^+) \) is a strong covering matrix for \( \lambda^+ \) if:

1. \( A(0, \alpha) \subseteq A(1, \alpha) \subseteq A(2, \alpha) \ldots \), for all \( \alpha \),
2. \( \bigcup_n A(n, \alpha) = \alpha \), for all \( \alpha \),
3. \( |A(n, \alpha)| < \lambda \), for all \( n \) and \( \alpha \),
4. for all \( \alpha < \beta \) there is \( n \) such that \( A(m, \alpha) \subseteq A(m, \beta) \), for all \( m \geq n \),
5. for all \( x \in \mathcal{P}_{\omega_1}(\lambda^+) \) there is \( \gamma x < \lambda^+ \) such that for all \( \alpha \geq \gamma x \) there is \( n \) such that \( A(m, \alpha) \cap x = A(m, \gamma x) \cap x \), for all \( m \geq n \).
**Proposition**

Assume $\lambda > 2^{\aleph_0}$ is of countable cofinality. Then there is a strong covering matrix $\mathcal{A}$ for $\lambda^+$.

**Proposition (Viale)**

Assume for all $\lambda > 2^{\aleph_0}$ of countable cofinality and a strong covering matrix $\mathcal{A}$ for $\lambda^+$, there is an unbounded set $B \subseteq \lambda^+$ such that $\mathcal{P}_{\omega_1}(B)$ is covered by $\mathcal{A}$. Then SCH holds.

**Remark**

$\mathcal{P}_{\omega_1}(B)$ is **covered** by $\mathcal{A}$ if, for every $x \in \mathcal{P}_{\omega_1}(B)$, there are $n, \alpha$ such that $x \subseteq A(n, \alpha)$. 
Lemma

Suppose $\text{cof}(\lambda) = \omega$ and $\mathcal{A}$ is a strong covering matrix for $\lambda^+$. Let $\theta$ be sufficiently large regular cardinal. Let $M < H_\theta$ be an $\omega_1$-guessing internally unbounded model of size $\aleph_1$. Let $\delta_M = \sup(M \cap \lambda^+)$. Then there is $n$ such that $A(m, \delta_M) \cap x \in M$, for all $x \in P_{\omega_1}(\lambda^+) \cap M$ and $m \geq n$.

Proof.

Otherwise, for each $n$, pick $x_n \in P_{\omega_1}(\lambda^+) \cap M$ with $A(n, \delta_M) \cap x_n \notin M$. By internal unboundedness of $M$ find countable $x \in M$ such that $\bigcup_n x_n \subseteq x$. By elementarity of $M$, $\gamma x \in M$. By definition of $\gamma x$ there is $n_0$ such that for all $n \geq n_0$

$$A(n, \delta_M) \cap x = A(n, \gamma x) \cap x \in M.$$

Given $n \geq n_0$ we have $A(n, \delta_M) \cap x = A(n, \gamma x) \cap x \in M$, and hence:

$$A(n, \delta_M) \cap x_n = A(n, \delta_M) \cap x \cap x_n \in M.$$

This is a contradiction.
**Theorem (Viale, Krueger)**

Assume $\text{GM}(\omega_2, \omega_1)$. Then SCH holds.

**Proof.**

Let $\lambda > 2^{\aleph_0}$ be of countable cofinality and let $\mathcal{A}$ be a strong covering matrix for $\lambda^+$. We find an unbounded $B \subseteq \lambda^+$ such that $\mathcal{P}_{\omega_1}(B)$ is covered by $\mathcal{A}$.

Fix $\theta$ large enough and an I.U. $\omega_1$-guessing model $M < H_\theta$ of size $\aleph_1$ with $\mathcal{A} \in M$. We may assume $\text{cof}(\delta_M) = \omega_1$. By previous lemma there is $n_0$ such that $A(m, \delta_M) \cap x \in M$, for all $x \in \mathcal{P}_{\omega_1}(\lambda^+) \cap M$, and all $m \geq n_0$.

Since $M$ is an $\omega_1$-guessing model we can find, for each $m \geq n_0$, $A_m \in M$ such that $A(m, \delta_M) \cap M = A_m \cap M$. Since $\text{cof}(\delta_M) = \omega_1$ we can find $m \geq n_0$ such that $A(m, \delta_M) \cap M$ is unbounded in $\delta_M$, but since $A_m \in M$ and $A(m, \delta_M) \cap M = A_m \cap M$, it follows that $A_m$ is unbounded in $\lambda^+$.

If $x \in \mathcal{P}_{\omega_1}(A_m) \cap M$ then $x$ is covered by $A(m, \delta_M)$. By elementarity of $M$ it follows that every $x \in \mathcal{P}_{\omega_1}(A_m)$ is covered by some member of $\mathcal{A}$. Hence, we can set $B = A_m$. 
Guessing models are closely related to the approachability ideal $I[\lambda]$.

**Definition**

Let $\lambda$ be a regular cardinal and $\bar{a} = (a_\xi : \xi < \lambda)$ a sequence of bounded subsets of $\lambda$. We let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal $c \subseteq \delta$ such that:

1. $\text{otp}(c) < \delta$, in particular $\delta$ is singular,
2. for all $\gamma < \delta$, there is $\eta < \delta$ such that $c \cap \gamma = a_\eta$.

**Definition (Shelah)**

Suppose $\lambda$ is regular. $I[\lambda]$ is the ideal generated by the sets $B(\bar{a})$, for sequences $\bar{a}$ as above, and the non stationary ideal $\text{NS}_\lambda$. 
This ideal was defined by Shelah in the late 1970s. \( I[\lambda] \) and its variations have been extensively studied over the past 40 years.

For regular \( \kappa < \lambda \) we let \( S^{\kappa}_\lambda = \{ \alpha < \lambda : \text{cof}(\alpha) = \kappa \} \).

**Theorem (Shelah)**

Suppose \( \lambda \) is a regular cardinal.

1. Then \( S^{<\lambda}_{\lambda^+} \in I[\lambda^+] \).
2. Suppose \( \kappa \) is regular and \( \kappa^+ < \lambda \). Then there is a stationary subset of \( S^{\kappa}_\lambda \) which belongs to \( I[\lambda] \).

The **approachability property** \( \text{AP}_{\kappa^+} \) states that \( \kappa^+ \in I[\kappa^+] \). For a regular cardinal \( \kappa \) the issue is to understand \( I[\kappa^+] \uparrow S^{\kappa}_{\kappa^+} \).
Approachability ideal

**Proposition**

Assume $GM(\kappa^+, \kappa)$. Then $\kappa^+ \notin I[\kappa^+]$.

**Proof.**

Let $\bar{a} = (a_\xi : \xi < \kappa^+)$ be a sequence of bounded subsets of $\kappa^+$.

Fix $M < H_{\kappa^+}$ a $\kappa$-guessing model of size $\kappa$ with $\bar{a} \in M$.

Let $\delta = M \cap \kappa^+$. We claim that $\delta \notin B(\bar{a})$.

Suppose $\delta \in B(\bar{a})$, and let $c \subseteq \delta$ witness this. Thus $\mu = o.t.(c) < \delta$.

For $\gamma < \delta$ there is $\eta < \delta$ such that $c \cap \gamma = a_\eta \in M$.

So $c$ is $\kappa$-approximated in $M$.

Since $M$ is $\kappa$-guessing model, there is $c^* \in M$ with $c = c^* \cap M$.

Then $c$ is an initial segment of $c^*$, and $c = c^*(\mu) \cap \kappa^+$, where $c^*(\mu)$ is the $\mu$-th element of $c^*$.

It follows that $c \in M$, and hence also $\delta = \sup(c) \in M$, a contradiction!
**Question (Shelah)**
Can $I[\omega_2] \upharpoonright S^{\omega_1}_{\omega_2}$ consistently be the nonstationary ideal on $S^{\omega_1}_{\omega_2}$?

**Theorem (Mitchell)**
Suppose $\kappa$ is $\kappa^+$-Mahlo. Then there is a generic extension in which $\kappa = \omega_2$ and $I[\omega_2] \upharpoonright S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$.

**Definition (Mitchell Property)**
For $\lambda$ regular, $MP(\lambda^+)$ denotes the statement that $I[\lambda^+] \upharpoonright S^{\lambda^+}_{\lambda^+}$ is the nonstationary ideal on $S^{\lambda^+}_{\lambda^+}$.

**Remark**
$MP(\omega_2)$ implies $2^{\kappa_0} \geq \kappa_3$. 
Some questions

Some questions:

1. Does $\text{GM}(\omega_2, \omega_1)$ imply $\text{MP}(\omega_2)$?
2. What about $\text{GM}(\omega_3, \omega_2)$?
3. Does $\text{GM}(\omega_2, \omega_1)$ bound the continuum?

Some answers:

1. No! $\text{GM}(\omega_2, \omega_1)$ is consistent with $\mathfrak{c} = \aleph_2$. (Viale–Weiss)
2. $\text{GM}(\omega_3, \omega_2)$ is consistent with CH. (Trang)
3. $\text{GM}(\omega_2, \omega_1)$ is consistent with continuum large. (Cox–Krueger)
Strong guessing models

**Idea:** combine $\text{GM}(\omega_2, \omega_1)$, $\text{GM}(\omega_3, \omega_2)$ and $\text{MP}(\omega_2)$.

**Definition**

Let $R$ be a model of a fragment of ZFC. We say that $M < R$ is a **strong $\omega_1$-guessing model** if $M$ can be written as the union of an increasing $\omega_1$-continuous $\in$-chain $(M_\xi : \xi < \omega_2)$ of $\omega_1$-guessing models of size $\omega_1$.

**Remark**

Every strongly $\omega_1$-guessing model is also an $\omega_1$-guessing model.

$$\mathcal{G}^+_{\omega_3, \omega_1}(R) = \{ M \in [R]^{\omega_2} : M \text{ is a strong } \omega_1\text{-guessing model} \}.$$

**Definition**

$\text{GM}^+(\omega_3, \omega_1)$ states that $\mathcal{G}^+_{\omega_3, \omega_1}(H_\theta)$ is stationary, for all large enough $\theta$. 
Theorem

$\text{GM}^+ (\omega_3, \omega_1)$ implies the following:

1. $\text{GM}(\omega_3, \omega_2)$ and $\text{GM}(\omega_2, \omega_1)$.
2. $\text{MP}(\omega_2)$ and hence $2^{\aleph_0} \geq \aleph_3$.
3. there are no weak $\omega_1$-Kurepa trees nor weak $\omega_2$-Kurepa trees.
4. the tree property at $\omega_2$ and $\omega_3$.
5. the failure of $\square(\lambda)$, for all $\lambda \geq \omega_2$.

Theorem (Mohammadmour, V.)

Assume there are two supercompact cardinals. There there is a generic extension in which $\text{GM}^+ (\omega_3, \omega_1)$ holds.
**Definition**

Suppose \((T, <)\) a tree of size and height \(\aleph_1\). \(T\) is **weakly special** if there is a function \(\sigma : T \to \omega\) such that if \(\sigma(r) = \sigma(s) = \sigma(t)\) with \(r < s, t\), then \(s\) and \(t\) are comparable.

**Proposition**

If \(T\) is a tree of height and size \(\omega_1\) and is weakly special then \(T\) has at most \(\aleph_1\) many cofinal branches.

**Proof.**

Let \(f\) be a weak specializing map of \(T\). If \(b\) is a cofinal branch there is an integer \(n_b\) such that \(|f^{-1}(n_b) \cap b| = \aleph_1\). Let \(t_b\) be the least element of \(f^{-1}(n_b) \cap b\). Then the map \(b \mapsto t_b\) is injective from the set of cofinal branches to \(T\).
Let $X$ be a set.

$$T_X = \{ (Z, f) : Z \in X \text{ is uncountable and } f : Z \cap X \to 2 \}.$$

**Definition**

Suppose that $M$ is an $\omega_1$-guessing model. Let $(M_\xi : \xi < \omega_1)$ be an IU-sequence. Let

$$T(M) = \bigcup_{\xi < \omega_1} (T_{M_\xi} \cap M).$$

Define the order $\leq$ on $T(M)$ be letting $(Z, f) \leq (W, g)$ if and only if $Z = W$ and $f \subseteq g$.

**Remark**

Suppose that $M$ is an $\omega_1$-guessing model of size $\aleph_1$. Then $(T(M), \leq)$ is a tree of size and height $\omega_1$ with at most $\aleph_1$ cofinal branches.

**Definition**

We say that $M$ is a **special guessing model** if $T(M)$ is weakly special.
Proposition

If $M$ is a special $\omega_1$-guessing model of size $\omega_1$ then $M$ remains $\omega_1$-guessing in any outer universe $W$ of $V$ with $\omega_1^W = \omega_1^V$.

Proof.

Suppose $W$ is an outer universe with $\omega_1^W = \omega_1^V$. Let $X \in M$ and suppose $f : X \to 2$ with $f \in W$ is $\omega_1$-approximated in $M$. Then $f$ gives a branch through $T(M)$. But all the branches of $T(M)$ are in $V$, hence $f \in V$. Since $M$ is $\omega_1$-guessing model in $V$, it follows that $f \in M$.  \qed
**Definition (SGM(ω₂, ω₁))**

SGM(ω₂, ω₁) denotes the statement that the set of special ω₁-guessing models of size ℵ₁ is stationary in [H₀]ℵ₁, for all sufficiently large regular θ.

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**Theorem (Cox-Krueger)**

- SGM(ω₂, ω₁) is consistent with continuum arbitrary large, modulo the existence of a supercompact cardinal.

- Assume SGM(ω₂, ω₁). Then Souslin’s Hypothesis holds.

- Assume SGM(ω₂, ω₁) and 2ℵ₀ < ℵω₁. Then the principle AMP(ω₁) holds: every forcing that adds a new subset of ω₁ either adds a real or collapsing some cardinal below 2ℵ₀.
Definition

A model $M$ of cardinality $\omega_2$ is **special strongly $\omega_1$-guessing** if it is the union of an $\epsilon$-increasing chain $(M_\xi : \xi < \omega_2)$ which is continuous at cofinality $\omega_1$ of special $\omega_1$-guessing models of cardinality $\omega_1$.

Definition (SGM$^+(\omega_3, \omega_1)$)

SGM$^+(\omega_3, \omega_1)$ states that the set of special strongly $\omega_1$-guessing models is stationary in $[H_\theta]^{\omega_2}$, for all large enough regular $\theta$. 
Theorem (Mohammadpour, V.)

Assume there are two supercompact cardinals. There is a generic extension in which $\text{SGM}^+ (\omega_3, \omega_1)$ holds.

Theorem (Mohammadpour, V.)

Assume $\text{SGM}^+ (\omega_3, \omega_1)$, $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_1} < \aleph_{\omega_2}$. Then $\text{AMP}(\omega_2)$ holds: every poset that adds a new subset of $\omega_2$ either adds a real or collapses some cardinal.
$2^{\aleph_0} \geq \aleph_3$

$\neg \Box (\omega_2, \lambda)$  
TP(\kappa_2)  
SCH  
MP(\omega_2)

AMP(\kappa_1)  
SH  
GM(\omega_2, \omega_1)  
TP(\kappa_3)  
FS(\omega_2, \omega_1)

AMP(\kappa_2)  
SGM(\omega_2, \omega_1)  
GM(\omega_2, \omega_1)  
GM^+(\omega_3, \omega_1)  
SFS(\omega_2, \omega_1)

SGM^+(\omega_3, \omega_1)
Thank You!