Strong guessing models

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Outline

Background and motivation

This is joint work with my PhD student **R. Mohammadpour**.

General form of forcing axioms. Let $\mathcal{K}$ be a class of forcing notions and κ an uncountable cardinal.

$\mathrm{FA}_\kappa(\mathcal{K})$

For every $\mathcal{P} \in \mathcal{K}$ and a family $\mathcal{D}$ of κ dense subsets of $\mathcal{P}$ there is a filter G in $\mathcal{P}$ such that $G \cap D \neq \varnothing$, for all $D \in \mathcal{D}$.

- $\mathrm{MA}_\kappa \equiv \mathrm{FA}_\kappa(\mathrm{ccc})$
- $\mathrm{PFA} \equiv \mathrm{FA}_{\aleph_1}(\mathrm{proper})$
- $\mathrm{MM} \equiv \mathrm{FA}_{\aleph_1}(\mathrm{stationary\ preserving})$

Remark

$\mathcal{K}$ cannot be the class of all posets or even all posets preserving $\aleph_1$.

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PFA implies

- $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$
- Singular Cardinal Hypothesis
- The tree property at $\aleph_2$
- the failure of $\square(\kappa)$, for regular $\kappa > \aleph_1$.

MM implies

- NS_{ω_1} is $\aleph_2$-saturated
- Chang's conjecture $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$

We are looking for higher forcing axioms that have similar structural consequences. In particular we want to have $2^{\aleph_0} > \aleph_2$.

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Guessing models

We first search for some principles that follow from PFA, imply most of its structural properties, but are consistent with $2^{\aleph_0}$ being bigger than ω_2. The key notion is that of a **guessing model**.

Definition (Viale)

Let R be a model of a fragment of set theory and $M \prec R$. Let γ be a cardinal. Let $Z \in M$ and $f : Z \to 2$ be a function.

1. *f is γ-**approximated** in M if $f \restriction C \in M$, for all $C \in \mathcal{P}_\gamma(Z) \cap M$.*
2. *f is **guessed** in M if there is $\overline{f} \in M$ such that $f \restriction M = \overline{f} \restriction M$.*

*We say that M is a γ-**guessing model** if every $f \in R$ which is γ-approximated in M is guessed in M.*

Remark

$M \prec H_\theta$ is a γ-guessing model iff the transitive collapse $\bar{M}$ of M has the γ-approximation property in the sense of Hamkins.

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Write $\mathcal{P}^*_\kappa(R)$ for the set of all $M \prec R$ such that $M \cap \kappa \in \kappa$. For $\gamma \leq \kappa$ we let

$$\mathfrak{G}_{\kappa,\gamma}(R) = \{M \in \mathcal{P}^*_\kappa(R) : M \text{ is } \gamma\text{-guessing}\}.$$

Definition (Viale)

$\mathrm{GM}(\kappa, \gamma)$ *is the statement that* $\mathfrak{G}_{\kappa,\gamma}(H_\theta)$ *is stationary, for all sufficiently large* θ.

We are primarily interested in $\gamma = \omega_1$ and $\kappa = \omega_2$, i.e. ω_1-guessing models of size ω_1.

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Lemma (Viale)

1. *If M is $\aleph_0$-guessing then $\kappa_M = M \cap \kappa$ and κ are inaccessible.*
2. *$M \prec V_\delta$ is $\aleph_0$-guessing iff $\bar{M} = V_{\bar{\delta}}$, for some $\bar{\delta}$, where $\bar{M}$ is the transitive collapse of M.*

The following is a reformulation of Magidor's characterization of supercompactness in terms of $\aleph_0$-guessing models.

Theorem (Magidor)

κ is supercompact iff $\mathrm{GM}(\kappa, \aleph_0)$ holds.

Remark

For this reason we use the term **Magidor models** for $\aleph_0$-guessing models.

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Theorem (Viale, Weiss)

PFA *implies* $\mathrm{GM}(\omega_2, \omega_1)$.

Theorem (Weiss)

$\mathrm{GM}(\omega_2, \omega_1)$ *implies*

1. *the failure of* $\square(\lambda)$, *for all regular* $\lambda \geq \omega_2$.
2. $\mathrm{TP}(\omega_2)$, *in fact,* $\mathrm{TP}(\omega_2, \lambda)$, *for* $\lambda \geq \omega_2$.

Theorem (Viale, Krueger)

$\mathrm{GM}(\omega_2, \omega_1)$ *implies* SCH.

Theorem (Cox, Krueger)

$\mathrm{GM}(\omega_2, \omega_1)$ *is consistent with* $2^{\aleph_0}$ *arbitrarily large.*

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Definition

Let $\theta > \omega_1$ be a regular cardinal. Let $N \prec H_\theta$ be of size $\aleph_1$.

- *We say that N is* ***internally unbounded (I.U.)*** *if there is an $\in$-chain of countable models $(N_\xi : \xi < \omega_1)$ such that $N = \bigcup_\xi N_\xi$.*
- *We say that N is* ***internally club (I.C.)*** *if the above sequence can be taken to be continuous.*

Definition

Let $M \prec H_\theta$ be of size $\aleph_1$. We say that M is ***locally internally unbounded*** *if $\mathcal{P}_{\omega_1}(X) \cap M$ is cofinal in $\mathcal{P}_{\omega_1}(X \cap M)$, for every $X \in M$.*

Fact

Suppose $\theta_0 < \theta_1$ are regular and $M \prec H_{\theta_1}$ is locally internally unbounded with $\theta_0 \in M$. Then $M \cap H_{\theta_0}$ is internally unbounded.

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Theorem (Krueger)

If $M \prec H_\theta$ is an ω_1-guessing model of size $\aleph_1$, then M is locally internally unbounded.

Proof.

Let $X \in M$ and $x \in \mathcal{P}_{\omega_1}(X \cap M)$. We need to find a countable $y \in M$ with $x \subseteq y$.
Let $f : \omega \to x$ be a bijection, and set $x_n = f"n$, hence $x_n \subseteq x_{n+1}$.
Let $A = \{x_n : n \in \omega\}$. Then $A \subseteq [X]^{<\omega} \in M$.

- If A is countably approximated in M, since M is an ω_1-guessing model, $A \in M$, and hence $x = \bigcup A \in M$. Set $y = x$.
- Otherwise there is a countable $Y \subseteq [X]^{<\omega}$ in M such that $A \cap Y \notin M$, but then $A \cap Y$ is infinite, and $x = \bigcup(A \cap Y) \subseteq \bigcup Y \in M$. Set $y = \bigcup Y$.

□

Definition (Viale)

Let λ be singular of cofinality ω. $\mathscr{A} = (A(n,\alpha) : n < \omega, \alpha < \lambda^+)$ is a ***strong covering matrix for*** *λ^+ if:*

1. $A(0,\alpha) \subseteq A(1,\alpha) \subseteq A(2,\alpha) \ldots$, *for all* α,
2. $\bigcup_n A(n,\alpha) = \alpha$, *for all* α,
3. $|A(n,\alpha)| < \lambda$, *for all* n *and* α,
4. *for all* $\alpha < \beta$ *there is* n *such that* $A(m,\alpha) \subseteq A(m,\beta)$, *for all* $m \geq n$,
5. *for all* $x \in \mathcal{P}_{\omega_1}(\lambda^+)$ *there is* $\gamma_x < \lambda^+$ *such that for all* $\alpha \geq \gamma_x$ *there is* n *such that* $A(m,\alpha) \cap x = A(m,\gamma_x) \cap x$, *for all* $m \geq n$.

Proposition

Assume $\lambda > 2^{\aleph_0}$ is of countable cofinality. Then there is a strong covering matrix $\mathscr{A}$ for λ^+.

Proposition (Viale)

Assume for all $\lambda > 2^{\aleph_0}$ of countable cofinality and a strong covering matrix $\mathscr{A}$ for λ^+, there is an unbounded set $B \subseteq \lambda^+$ such that $\mathcal{P}_{\omega_1}(B)$ is covered by $\mathscr{A}$. Then SCH *holds.*

Remark

$\mathcal{P}_{\omega_1}(B)$ is **covered** by $\mathscr{A}$ if, for every $x \in \mathcal{P}_{\omega_1}(B)$, there are n, α such that $x \subseteq A(n, \alpha)$.

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Lemma

Suppose $\operatorname{cof}(\lambda) = \omega$ *and* $\mathscr{A}$ *is a strong covering matrix for* λ^+*. Let* θ *be sufficiently large regular cardinal. Let* $M \prec H_\theta$ *be an* ω_1*-guessing internally unbounded model of size* $\aleph_1$*. Let* $\delta_M = \sup(M \cap \lambda^+)$*. Then there is* n *such that* $A(m, \delta_M) \cap x \in M$*, for all* $x \in \mathcal{P}_{\omega_1}(\lambda^+) \cap M$ *and* $m \geq n$*.*

Proof.

Otherwise, for each n, pick $x_n \in \mathcal{P}_{\omega_1}(\lambda^+) \cap M$ with $A(n, \delta_M) \cap x_n \notin M$.
By internal unboundedness of M find countable $x \in M$ such that $\bigcup_n x_n \subseteq x$.
By elementarity of M, $\gamma_x \in M$.
By definition of γ_x there is n_0 such that for all $n \geq n_0$

$$A(n, \delta_M) \cap x = A(n, \gamma_x) \cap x \in M.$$

Given $n \geq n_0$ we have $A(n, \delta_M) \cap x = A(n, \gamma_x) \cap x \in M$, and hence:

$$A(n, \delta_M) \cap x_n = A(n, \delta_M) \cap x \cap x_n \in M.$$

This is a contradiction.

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Theorem (Viale, Krueger)

Assume $\mathrm{GM}(\omega_2, \omega_1)$. *Then* SCH *holds.*

Proof.

Let $\lambda > 2^{\aleph_0}$ be of countable cofinality and let $\mathscr{A}$ be a strong covering matrix for λ^+. We find an unbounded $B \subseteq \lambda^+$ such that $\mathcal{P}_{\omega_1}(B)$ is covered by $\mathscr{A}$.

Fix θ large enough and an I.U. ω_1-guessing model $M \prec H_\theta$ of size $\aleph_1$ with $\mathscr{A} \in M$. We may assume $\mathrm{cof}(\delta_M) = \omega_1$. By previous lemma there is n_0 such that $A(m, \delta_M) \cap x \in M$, for all $x \in \mathcal{P}_{\omega_1}(\lambda^+) \cap M$, and all $m \geq n_0$.

Since M is an ω_1-guessing model we can find, for each $m \geq n_0$, $A_m \in M$ such that $A(m, \delta_M) \cap M = A_m \cap M$. Since $\mathrm{cof}(\delta_M) = \omega_1$ we can find $m \geq n_0$ such that $A(m, \delta_M) \cap M$ is unbounded in δ_M, but since $A_m \in M$ and $A(m, \delta_M) \cap M = A_m \cap M$, it follows that A_m is unbounded in λ^+.

If $x \in \mathcal{P}_{\omega_1}(A_m) \cap M$ then x is covered by $A(m, \delta_M)$. By elementarity of M it follows that every $x \in \mathcal{P}_{\omega_1}(A_m)$ is covered by some member of $\mathscr{A}$. Hence, we can set $B = A_m$. □

Approachability ideal

Guessing models are closely related to the approachability ideal $I[\lambda]$.

Definition

Let λ be a regular cardinal and $\bar{a} = (a_\xi : \xi < \lambda)$ a sequence of bounded subsets of λ. We let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal $c \subseteq \delta$ such that:

1. $\mathrm{otp}(c) < \delta$*, in particular δ is singular,*
2. *for all $\gamma < \delta$, there is $\eta < \delta$ such that $c \cap \gamma = a_\eta$.*

Definition (Shelah)

Suppose λ is regular. $I[\lambda]$ is the ideal generated by the sets $B(\bar{a})$, for sequences $\bar{a}$ as above, and the non stationary ideal NS_λ.

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Approachability ideal

This ideal was defined by Shelah in the late 1970s. $I[\lambda]$ and its variations have been extensively studied over the past 40 years.

For regular $\kappa < \lambda$ we let $S^{\kappa}_{\lambda} = \{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$.

Theorem (Shelah)

Suppose λ is a regular cardinal.

1. *Then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$.*
2. *Suppose κ is regular and $\kappa^+ < \lambda$. Then there is a stationary subset of S^{κ}_{λ} which belongs to $I[\lambda]$.*

The **approachability property** AP_{κ^+} states that $\kappa^+ \in I[\kappa^+]$. For a regular cardinal κ the issue is to understand $I[\kappa^+] \restriction S^{\kappa}_{\kappa^+}$.

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Approachability ideal

Proposition

Assume $\mathrm{GM}(\kappa^+, \kappa)$. *Then* $\kappa^+ \notin I[\kappa^+]$.

Proof.

Let $\vec{a} = (a_\xi : \xi < \kappa^+)$ be a sequence of bounded subsets of κ^+.
Fix $M \prec H_{\kappa^{++}}$ a κ-guessing model of size κ with $\vec{a} \in M$.
Let $\delta = M \cap \kappa^+$. We claim that $\delta \notin B(\vec{a})$.
Suppose $\delta \in B(\vec{a})$, and let $c \subseteq \delta$ witness this. Thus $\mu = \mathrm{o.t.}(c) < \delta$.
For $\gamma < \delta$ there is $\eta < \delta$ such that $c \cap \gamma = a_\eta \in M$.
So c is κ-approximated in M.
Since M is κ-guessing model, there is $c^* \in M$ with $c = c^* \cap M$.
Then c is an initial segment of c^*, and $c = c^*(\mu) \cap \kappa^+$, where $c^*(\mu)$ is the μ-th element of c^*.
It follows that $c \in M$, and hence also $\delta = \sup(c) \in M$, a contradiction!

$\square$

Question (Shelah)

Can $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ consistently be the nonstationary ideal on $S^{\omega_1}_{\omega_2}$?

Theorem (Mitchell)

Suppose κ is κ^+-Mahlo. Then there is a generic extension in which $\kappa = \omega_2$ and $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$.

Definition (Mitchell Property)

For λ regular, $\mathrm{MP}(\lambda^+)$ denotes the statement that $I[\lambda^+] \restriction S^{\lambda}_{\lambda^+}$ is the nonstationary ideal on $S^{\lambda}_{\lambda^+}$.

Remark

$\mathrm{MP}(\omega_2)$ implies $2^{\aleph_0} \geq \aleph_3$.

Some questions

Some questions:

1. Does $\mathrm{GM}(\omega_2, \omega_1)$ imply $\mathrm{MP}(\omega_2)$?
2. What about $\mathrm{GM}(\omega_3, \omega_2)$?
3. Does $\mathrm{GM}(\omega_2, \omega_1)$ bound the continuum?

Some answers:

1. No! $\mathrm{GM}(\omega_2, \omega_1)$ is consistent with $\mathfrak{c} = \aleph_2$. (Viale–Weiss)
2. $\mathrm{GM}(\omega_3, \omega_2)$ is consistent with CH. (Trang)
3. $\mathrm{GM}(\omega_2, \omega_1)$ is consistent with continuum large.(Cox– Krueger)

Strong guessing models

Idea: combine $\mathrm{GM}(\omega_2, \omega_1)$, $\mathrm{GM}(\omega_3, \omega_2)$ and $\mathrm{MP}(\omega_2)$.

Definition

*Let R be a model of a fragment of ZFC. We say that $M \prec R$ is a **strong ω_1-guessing model** if M can be written as the union of an increasing ω_1-continuous $\in$-chain $(M_\xi : \xi < \omega_2)$ of ω_1-guessing models of size ω_1.*

Remark

Every strongly ω_1-guessing model is also an ω_1-guessing model.

$$\mathfrak{G}^+_{\omega_3,\omega_1}(R) = \{M \in [R]^{\omega_2} : M \text{ is a strong } \omega_1\text{-guessing model}\}.$$

Definition

$\mathrm{GM}^+(\omega_3, \omega_1)$ *states that* $\mathfrak{G}^+_{\omega_3,\omega_1}(H_\theta)$ *is stationary, for all large enough* θ.

Strong guessing models

Theorem

$\mathrm{GM}^+(\omega_3, \omega_1)$ *implies the following:*

1. $\mathrm{GM}(\omega_3, \omega_2)$ *and* $\mathrm{GM}(\omega_2, \omega_1)$.
2. $\mathrm{MP}(\omega_2)$ *and hence* $2^{\aleph_0} \geq \aleph_3$.
3. *there are no weak* ω_1*-Kurepa trees nor weak* ω_2*-Kurepa trees.*
4. *the tree property at* ω_2 *and* ω_3.
5. *the failure of* $\square(\lambda)$*, for all* $\lambda \geq \omega_2$.
6. *Singular Cardinal Hypothesis.*

Theorem (Mohammadpour, V.)

Assume there are two supercompact cardinals. There there is a generic extension in which $\mathrm{GM}^+(\omega_3, \omega_1)$ *holds.*

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Special guessing models

Definition

Suppose $(T, <)$ a tree of size and height $\aleph_1$. T is ***weakly special*** *if there is a function $\sigma : T \to \omega$ such that if $\sigma(r) = \sigma(s) = \sigma(t)$ with $r < s, t$, then s and t are comparable.*

Proposition

If T is a tree of height and size ω_1 and is weakly special then T has at most $\aleph_1$ many cofinal branches.

Proof.

Let f be a weak specializing map of T. If b is a cofinal branch there is an integer n_b such that $|f^{-1}(n_b) \cap b| = \aleph_1$. Let t_b be the least element of $f^{-1}(n_b) \cap b$. Then the map $b \mapsto t_b$ is injective from the set of cofinal branches to T. $\square$

Let X be a set.

$$T_X = \{(Z, f) : Z \in X \text{ is uncountable and } f : Z \cap X \to 2\}.$$

Definition

Suppose that M is an ω_1-guessing model. Let $(M_\xi : \xi < \omega_1)$ be an IU*-sequence. Let*

$$T(M) = \bigcup_{\xi<\omega_1} (T_{M_\xi} \cap M).$$

Define the order $\leq$ on $T(M)$ be letting $(Z, f) \leq (W, g)$ if and only if $Z = W$ and $f \subseteq g$.

Remark

Suppose that M is an ω_1-guessing model of size $\aleph_1$. Then $(T(M), \leq)$ is a tree of size and height ω_1 with at most $\aleph_1$ cofinal branches.

Definition

We say that M is a ***special guessing model*** *if $T(M)$ is weakly special.*

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Proposition

If M is a special ω_1-guessing model of size ω_1 then M remains ω_1-guessing in any outer universe W of V with $\omega_1^W = \omega_1^V$.

Proof.

Suppose W is an outer universe with $\omega_1^W = \omega_1^V$. Let $X \in M$ and suppose $f : X \to 2$ with $f \in W$ is ω_1-approximated in M. Then f gives a branch through $T(M)$. But all the branches of $T(M)$ are in V, hence $f \in V$. Since M is ω_1-guessing model in V, it follows that $f \in M$. $\square$

Definition ($\mathrm{SGM}(\omega_2, \omega_1)$)

$\mathrm{SGM}(\omega_2, \omega_1)$ *denotes the statement that the set of special* ω_1*-guessing models of size* $\aleph_1$ *is stationary in* $[H_\theta]^{\aleph_1}$*, for all sufficiently large regular* θ*.*

Theorem (Cox-Krueger)

- $\mathrm{SGM}(\omega_2, \omega_1)$ *is consistent with continuum arbitrary large, modulo the existence of a supercompact cardinal.*
- *Assume* $\mathrm{SGM}(\omega_2, \omega_1)$*. Then Souslin's Hypothesis holds.*
- *Assume* $\mathrm{SGM}(\omega_2, \omega_1)$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$*. Then the principle* $\mathrm{AMP}(\omega_1)$ *holds: every forcing that adds a new subset of* ω_1 *either adds a real or collapsing some cardinal below* $2^{\aleph_0}$*.*

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Definition

*A model M of cardinality ω_2 is **special strongly ω_1-guessing** if it is the union of an $\in$-increasing chain $(M_\xi : \xi < \omega_2)$ which is continuous at cofinality ω_1 of special ω_1-guessing models of cardinality ω_1.*

Definition $(\mathrm{SGM}^+(\omega_3, \omega_1))$

$\mathrm{SGM}^+(\omega_3, \omega_1)$ *states that the set of special strongly ω_1-guessing models is stationary in $[H_\theta]^{\omega_2}$, for all large enough regular θ.*

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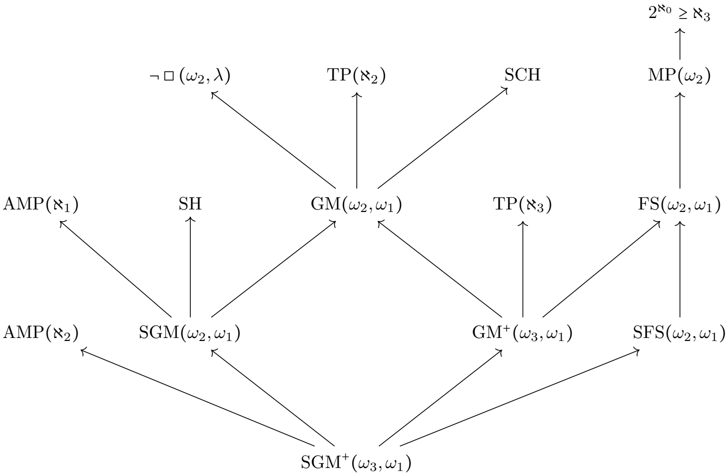
Theorem (Mohammadpour, V.)

Assume there are two supercompact cardinals. There there is a generic extension in which $\mathrm{SGM}^+(\omega_3, \omega_1)$ *holds.*

Theorem (Mohammadpour, V.)

Assume $\mathrm{SGM}^+(\omega_3, \omega_1)$, $2^{\aleph_0} < \aleph_{\omega_1}$ *and* $2^{\aleph_1} < \aleph_{\omega_2}$. *Then* $\mathrm{AMP}(\omega_2)$ *holds: every poset that adss a new subset of* ω_2 *either adds a real or collapses some cardinal.*



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Thank You!

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