Non-standard Models of Arithmetic and Their Standard Systems

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Standard Systems

Let $M \models \mathrm{PA}$ be non-standard. Every $a \in M$ can be identified as a binary sequence, let $(a)_i$ be the i-th bit.

The standard system of M is the following set of subsets of $\mathbb{N}$

$$\mathrm{SSy}(M) = \{\{i \in \mathbb{N} : M \models (a)_i = 1\} : a \in M - \mathbb{N}\}.$$

We say that the set $\{i \in \mathbb{N} : M \models (a)_i = 1\}$ is coded by a in M.

But there are other ways to present $\mathrm{SSy}(M)$. Let p_i denote the i-th prime number and let $p_i | a$ denote the formula saying that p_i divides a. Then

$$\begin{aligned}\mathrm{SSy}(M) &= \{\{i \in \mathbb{N} : M \models p_i | a\} : a \in M - \mathbb{N}\} \\ &= \{A \cap \mathbb{N} : A \text{ is definable in } M\}.\end{aligned}$$

Computability of Standard Systems

Theorem (Scott, 1962)

If M is a non-standard model then $\mathrm{SSy}(M)$ satisfies the following.

1. *If $Y \leq_T X \in \mathrm{SSy}(M)$ then $Y \in \mathrm{SSy}(M)$;*
2. *If X and Y are both in $\mathrm{SSy}(M)$ then*

$$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\} \in \mathrm{SSy}(M);$$

3. *If T is an infinite binary tree computable in $X \in \mathrm{SSy}(M)$ then there exists $Y \in \mathrm{SSy}(M)$ whose characteristic function as a countable binary sequence is an infinite path on T (we write $Y \in [T]$).*

A Turing ideal is a set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ satisfying (1-2) above, and a Scott set is a Turing ideal satisfying also (3). So $\mathrm{SSy}(M)$ is a Scott set for every non-standard model M.

Countable Scott Sets as Standard Systems

Theorem (Scott, 1962)

For each countable Scott set $\mathcal{S}$ and each consistent extension Γ of PA in $\mathcal{S}$, there exists a non-standard $M \models \Gamma$ s.t. $\mathrm{SSy}(M) = \mathcal{S}$.

Given Γ, to construct a completion Λ of Γ with Henkin's property is to find an infinite path on a Γ-computable binary tree. Since $\Gamma \in \mathcal{S}$, there is such a $\Lambda \in \mathcal{S}$ and thus a model $M_0 \models \Gamma$ with Λ being its elementary diagram in $\mathcal{S}$.

Then $\mathrm{SSy}(M_0) \subset \mathcal{S}$. If $X \in \mathcal{S} - \mathrm{SSy}(M_0)$, then let

$$\Gamma_X = \text{ The elementary diagram of } M_0 \\ \cup \{(c)_i = 1 : i \in X\} \cup \{(c)_i = 0 : i \notin X\},$$

which is consistent and in $\mathcal{S}$. By similar construction, there exists $M_1 \models \Gamma_X$, so $M_0 \prec M_1$ and $X \in \mathrm{SSy}(M_1) \subset \mathcal{S}$, moreover the elementary diagram of M_1 is an element of $\mathcal{S}$.

So we can have $M_0 \prec M_1 \prec \ldots$ and finally $M = \bigcup_n M_n$ is as desired.

The Scott Set Problem: Uncountable Scott Sets?

Question

Does every Scott set (countable or uncountable) equal to the standard system of some non-standard model of PA*?*

The Scott Set Problem Under CH

Theorem (Knight and Nadel, 1982)

Every Scott set of cardinality ω_1 *equals* $\mathrm{SSy}(M)$ *for some non-standard* $M \models \mathrm{PA}$.

So, under ZF + CH, the Scott set problem has an affirmative answer.

There are several known proofs of this theorem, most of them use recursively saturated models.

The Scott Set Problem Under CH

Recursively Saturated Models

Let M be a model and $A \subseteq M$. Let M_A be the expansion of M with just new constants naming elements of A.

A type of M over A is a type of M_A.

If $p(\vec{x})$ is a type of M over A, $\vec{x}$ and A being finite, and

$$\{\varphi(\vec{x}, \vec{y}) : \varphi(\vec{x}, \vec{a}) \in p\} \leq_T Z$$

for $Z \subseteq \mathbb{N}$, then p is a Z-recursive type. And p is recursive in $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$, iff p is recursive in some $Z \in \mathcal{S}$.

M is $\mathcal{S}$-recursively saturated, iff every $\mathcal{S}$-recursive $p(\vec{x})$ of M over some finite $\vec{a}$ from M is realized in M, i.e., there exists $\vec{b}$ in M s.t. $M \models \varphi(\vec{b}, \vec{a})$ for all $\varphi(\vec{x}, \vec{a}) \in p$.

Recursive means recursive in $\{\emptyset\}$.

The Scott Set Problem Under CH

The Proof of Knight and Nadel

The original proof has two parts:

1. Nadel proved that if M is a recursively saturated model of $\mathrm{Pr}' = \mathrm{Th}(\mathbb{Z}, +, 1)$ and $|M| = \omega_1$ then M can be expanded to a recursively saturated model N of PA. Clearly, $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.
2. Knight and Nadel proved that every Scott set is the standard system of some $M \models \mathrm{Pr}'$.

The Scott Set Problem Under CH

A Recent Proof via Friedman's Embedding Theorem

In 2015, Alf Dolich, Julia Knight, Karen Lange and David Marker gave a new proof, which needs the following properties of recursively saturated models.

Proposition

Every countable model has a recursively saturated elementary extension.

Theorem (H. Friedman's Embedding Theorem, 1973)

Let M and N be countable recursively saturated models of PA. If M and N are elementarily equivalent then

$$\mathrm{SSy}(M) \subseteq \mathrm{SSy}(N) \Leftrightarrow M \preceq N.$$

The proof goes as follows.

Let $\mathcal{S}$ be a Scott set of cardinality ω_1. Pick a consistent completion $T \in \mathcal{S}$ of PA. We can find countable Scott sets $(\mathcal{S}_\alpha : \alpha < \omega_1)$ s.t.

$$T \in \mathcal{S}_0 \subseteq \mathcal{S}_\alpha \subset \mathcal{S}_\beta \ (\alpha < \beta),\ \mathcal{S} = \bigcup_\alpha \mathcal{S}_\alpha.$$

By Scott's Theorem, for each α, let $M_\alpha \models T$ be countable, recursively saturated with $\mathrm{SSy}(M_\alpha) = \mathcal{S}_\alpha$.

By inductive applications of Friedman's Embedding Theorem, there exist elementary embeddings $f_{\alpha,\beta} : M_\alpha \to M_\beta$ for $\alpha < \beta < \omega_1$, s.t.

$$\alpha < \beta < \gamma \to f_{\alpha,\gamma} = f_{\beta,\gamma} \circ f_{\alpha,\beta}.$$

The limit of this elementary chain is a desired model.

Lemma (Ehrenfeucht)

Suppose that $\mathcal{S}$ is a Scott set, $X \in \mathcal{S}$ and $M \models \mathrm{PA}$ is countable with $\mathrm{SSy}(M) \subseteq \mathcal{S}$. Then there exists N s.t. $M \prec N$, $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

Note: it could be that neither the theory nor the elementary diagram of M is in $\mathcal{S}$.

It is not hard to see that Ehrenfeucht's Lemma implies the above theorem of Knight and Nadel.

And it happens that Ehrenfeucht's Lemma could also be proved by applications of Friedman's Embedding Lemma. (see Gitman, 2008)

But I shall present another proof of this lemma here.

The Scott Set Problem Under CH

Another Proof via A Lemma of Ehrenfeucht

Let $\mathcal{S}$ be a Scott set, $M \models \mathrm{PA}$ be countable with $\mathrm{SSy}(M) \subseteq \mathcal{S}$, $X \in \mathcal{S}$. We need N s.t. $M \prec N$, $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

It suffices to build a type $p(x)$ in M, s.t., if b realizes $p(x)$ (in an extension of M) then

(c1) b codes X;

(c2) for each function $F(x)$ definable in M with parameters, $F(b)$ codes a set in $\mathcal{S}$ (NEVER GO OUTSIDE).

The Scott Set Problem Under CH

Another Proof via A Lemma of Ehrenfeucht

Fix $a \in M - \mathbb{N}$. Our desired type will have realizations whose binary expansions are of length a.

We begin with $p_0(x)$ consisting of formulas below

$$x \in 2^a,\ x(i) = X(i),$$

where 2^a denotes the set of binay sequences of length a, $X(i) = 1$ if $i \in X$ and $X(i) = 0$ if $i \notin X$.

As a is an 'infinite natural number', $p_0(x)$ is finitely realizable in M.

If b realizes a type $p(x) \supseteq p_0(x)$ then b satisfies (c1) (i.e., it codes X).

Also note: $p_0(x)$ is recursive in X.

The Scott Set Problem Under CH

Another Proof via A Lemma of Ehrenfeucht

Let $F(x)$ be a function definable in M (with parameters). We need to ensure if b realizes our final type $p(x)$ then $F(b)$ codes something in $\mathcal{S}$.

Consider the following tree $T \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$: $(\sigma, \tau) \in T$, iff $|\sigma| = |\tau|$ ($|\cdot|$ being the length), and the following subset of M is non-empty

$$\{x \in 2^a : \forall i < |\sigma| \, (\sigma(i) = x(i) \wedge \tau(i) = (F(x))_i)\} .$$

Then, $T \in \mathrm{SSy}(M)$ and T is infinite.

$T \oplus X$ computes another infinite tree: $\tau \in T^X$ iff $(X \upharpoonright |\tau|, \tau) \in T$.

So, T^X has an infinite path $Y \in \mathcal{S}$. Let $p_1(x)$ be

$$p_0(x) \cup \{(F(x))_i = Y(i) : i \in \mathbb{N}\}.$$

So $p_1(x)$ is a type and if b realizes $p_1(x)$ then $F(b)$ codes Y.

Inductively, we can construct $p(x) = \bigcup_n p_n(x)$ as desired.

Scott Set Problem without CH

But the problem is still open if CH is not assumed.

Question

If CH fails, does the Scott Set Problem still have a positive answer? What if CH is replaced by some forcing axiom?

Ehrenfeucht Principle

A possible approach would be to generalize Ehrenfeucht's Lemma.

Definition (Gitman, 2008)

Let $\kappa \leq 2^\omega$ be a cardinal. The κ-Ehrenfeucht Principle is the statement that, if $M \models \mathrm{PA}$ is non-standard with $|M| < \kappa$ and $\mathrm{SSy}(M)$ is a subset of a Scott set $\mathcal{S}$ and $X \in \mathcal{S}$, then there exists N s.t. $M \prec N$ and $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

Clearly, the κ-Ehrenfeucht Principle implies that every Scott set of cardinality at most κ is $\mathrm{SSy}(M)$ for some $M \models \mathrm{PA}$.

An Obstacle

One may try to prove the ω_2-Ehrenfeucht Principle as follows. Given a Scott set $\mathcal{S}$ and $M \models \mathrm{PA}$ s.t. $|M| = \omega_1$ and $\mathrm{SSy}(M) \subseteq \mathcal{S}$, and $X \in \mathcal{S}$. Write M as an elementary chain of countable submodels M_α's. Then construct the following commutative diagram with $X \in \mathrm{SSy}(N_0)$ and $\mathrm{SSy}(N_\alpha) \subseteq \mathcal{S}$,

$$\begin{tikzcd}
N_0 \arrow[r, "\prec"] & N_1 \arrow[r, "\prec"] & \cdots \arrow[r, "\prec"] & N_\alpha \arrow[r, "\prec"] & \cdots \\
M_0 \arrow[u, "\prec"] \arrow[r, "\prec"] & M_1 \arrow[u, "\prec"] \arrow[r, "\prec"] & \cdots \arrow[r, "\prec"] & M_\alpha \arrow[u, "\prec"] \arrow[r, "\prec"] & \cdots
\end{tikzcd}$$

But in general, even the first square could be a mission impossible.

Proposition (Knight, in Knight and Nadel 1982)

For any countable non-standard $M_0 \models \mathrm{PA}$ and any $Y \subseteq \mathbb{N}$, there are M_1, N_0 s.t. $M_0 \prec M_1$, $M_0 \prec N_1$, $\mathrm{SSy}(M_1) = \mathrm{SSy}(N_0)$, and if N_1 is any model as in the above diagram then $Y \in \mathrm{SSy}(N_1)$.

Assuming Proper Forcing Axiom

A Scott set $\mathcal{S}$ is arithmetically closed, iff $X \in \mathcal{S}$ and $Y \in \Sigma_n^X$ imply $Y \in \mathcal{S}$.

A Scott set $\mathcal{S}$ is proper, iff the quotient $(\mathcal{S}, \subset)/\operatorname{Fin}$ (Fin is the ideal of finite subsets of $\mathbb{N}$) is a proper forcing poset.

Theorem (Victoria Gitman)

Assume PFA. The ω_2-Ehrenfeucht Principle holds for Scott sets which are arithmetically closed and proper, so every such Scott set is some $\mathrm{SSy}(M)$.

However, it is unknown whether there exists a non-trivial ($\neq \mathcal{P}(\mathbb{N})$) uncountable Scott set which is arithmetically closed and proper.

A Weaker Question

Question

(ZFC) Are there non-trivial standard systems of cardinality 2^ω?

Exercise

In ZF, show that there exist non-trivial Scott sets of cardinality 2^ω.

Hint: use Cohen forcing only for arithmetic formulas; and note that every arithmetic closed subset of $2^{\mathbb{N}}$ is a Scott set.

Models with Non-trivial Standard Systems

Theorem

In ZF. For every countable non-standard $M \models \mathrm{PA}$, there exists a family $(M_X : X \subset 2^{\mathbb{N}})$ s.t. $M = M_\emptyset \prec M_X$, $|M_X| = |\operatorname{SSy}(M_X)| = \max\{\omega, |X|\}$ and

$$X \subseteq Y \Leftrightarrow M_X \preceq M_Y \Leftrightarrow \operatorname{SSy}(M_X) \subseteq \operatorname{SSy}(M_Y).$$

Given M, we construct a type p in continuumly many variables $(x_f : f \in 2^\omega)$ s.t.

1. There exists $a \in M - \mathbb{N}$ s.t. $p \vdash x_f \in 2^a$;
2. If $f_0, f_1, \ldots, f_n \in 2^\omega$ are distinct and F is a function definable in M with parameters then there exists $i \in \mathbb{N}$ s.t.

$$p \vdash (x_{f_0})_i \neq (F(x_{f_1}, \ldots, x_{f_n}))_i.$$

Two Incomparable Extensions of $\mathrm{SSy}(M)$

Consider a baby case: let $M \models \mathrm{PA}$ be countable and non-standard, construct M_1, M_2 s.t. $M \prec M_i$ and $\mathrm{SSy}(M_i) \not\subseteq \mathrm{SSy}(M_j)$.

Pick $a \in M - \mathbb{N}$, we construct a type $p(x_1, x_2)$ and then form $M_i = M\langle x_i \rangle$.

We construct p as a union of $\bigcup_{s \in \mathbb{N}} p_s$, where each p_s is a finite type, $p_0 = \{x_1 \in 2^a, x_2 \in 2^a\}$, $p_s \subseteq p_{s+1}$, and there exists a positive $r \in \mathbb{Q}$ s.t.

$$M \models |p_s(M)| \geq r|p_0(M)| = r(2^a)^2,$$

where $p_s(M)$ is the subset of M defined by the finite type p_s.

Given p_s and an M-definable function F, we want to have p_{s+1} implying that $(x_1)_k \neq (F(x_2))_k$ for some $k \in \mathbb{N}$. For $m \in \mathbb{N}$, let

$$q = p_s \cup \{\forall k < m((x_1)_k = (F(x_2))_k)\}.$$

In M, $|q(M)| \leq 2^{a-m}2^a = 2^{-m}(2^a)^2$. So there exists $k \in \mathbb{N}$, s.t., $p_{s+1} = p_s \cup \{(x_1)_k \neq (F(x_2))_k\}$ can serve our purpose.

Many Incomparable Extensions of $\mathrm{SSy}(M)$

Now, we construct a desired type p in $(x_f : f \in 2^\omega)$. To construct p, we construct approximations p_i's to p, s.t.

1. Each p_i is a finite type in $(y_\sigma : \sigma \in 2^{n_i})$ for some n_i;
2. $p_i \vdash y_\sigma \in 2^a$ where $a \in M - \mathbb{N}$ is fixed;
3. Each i corresponds to a positive $r \in \mathbb{Q}$ s.t.
$$M \models |p_i(M)| > r(2^a)^{2^{n_i}};$$
4. If $i < j$ then $n_i < n_j$; moreover, if $\phi(y_{\sigma_1}, \ldots, y_{\sigma_k}) \in p_i$ and $\sigma_\ell \prec \tau_\ell \in 2^{n_j}$ then $\phi(y_{\tau_1}, \ldots, y_{\tau_k}) \in p_j$, where $\phi(y_{\tau_1}, \ldots, y_{\tau_k})$ is obtained from substituting y_{σ_ℓ} in ϕ by y_{τ_ℓ}.

By 4., if we let p be the set of $\phi(x_{f_1}, \ldots, x_{f_k})$'s s.t. $\phi(y_{\sigma_1}, \ldots, y_{\sigma_k}) \in p_i$ for some i and σ_n's prefixes of f_n, then p is a type.

Many Incomparable Extensions of $\mathrm{SSy}(M)$

For the incomparability, the p_i's should also be s.t.

5. If F is a k-ary definable function in M, $\sigma_\ell \in 2^{n_i}$ $(0 \le \ell \le k)$ are distinct, then there exists $j > i$, s.t., each tuple $(\tau_\ell : 0 \le \ell \le k)$ with $\sigma_\ell \prec \tau_\ell \in 2^{n_j}$ $(\ell = 0, \dots, k)$, corresponds to some $m \in \mathbb{N}$ with

$$p_j \vdash (y_{\tau_0})_m \neq (F(y_{\tau_1}, \dots, y_{\tau_k}))_m.$$

So, if p is constructed from p_i's as above, and if $f_0, \dots, f_k$ are distinct, choose i s.t. $f_\ell \restriction n_i$ are distinct.

Then apply 5., we get $j > i$ with $p_j \vdash (y_{\tau_0})_m \neq (F(y_{\tau_1}, \dots, y_{\tau_k}))_m$, where τ_ℓ's are prefixes of f_ℓ's in 2^{n_j}. So

$$p \vdash (x_{f_0})_m \neq (F(x_{f_1}, \dots, x_{f_k}))_m.$$

The construction of p_i's is not very different from the baby case.

Many Incomparable Extensions of $\mathrm{SSy}(M)$

Once we have the type p in $(x_f : f \in 2^{\mathbb{N}})$, we can find realizations $(b_f : f \in 2^{\mathbb{N}})$ is some big elementary extension N of M.

For each $X \subseteq 2^{\mathbb{N}}$, in N compute the Skolem hull of M and $(b_f : f \in X)$, let the resulted model be M_X.

So we have the models as desired.

Exercises

Exercise

1. *Fix a countable* $M \models \mathrm{PA}$ *and* $(f_n \in 2^{\mathbb{N}} : n \in \mathbb{N})$ *s.t.* $f_n \notin \mathrm{SSy}(M)$*, construct* $N \succ M$ *s.t.* $|\mathrm{SSy}(N)| = 2^\omega$ *and* $f_n \notin \mathrm{SSy}(N)$ *for all* n*.*
2. *Assume Martin's Axiom (*MA*), solve the above for* $|M| < 2^\omega$ *and* $(f_\alpha : \alpha < \kappa < 2^\omega)$*.*
3. *Assume* MA*, fix a countable* $M \models \mathrm{PA}$*,* $(g_n \in 2^{\mathbb{N}} : n \in \mathbb{N})$*,* $\kappa < 2^\omega$ *and* $(f_\alpha \in 2^{\mathbb{N}} : \alpha < \kappa)$*, s.t., the Turing ideal generated by* $\mathrm{SSy}(M)$ *and* g_n*'s does not contain any of* f_α*, construct* N *s.t.* $M \prec N$*,* $g_n \in \mathrm{SSy}(N)$*,* $f_\alpha \notin \mathrm{SSy}(N)$ *and* $|\mathrm{SSy}(N)| = 2^\omega$*.*

The above can be regarded as some approximations to the Scott Set Problem.

A Question

In the last exercise above, can we solve it for uncountable M and uncountably many g's?

More precisely, suppose that $M \models \mathrm{PA}$, $\omega < |M| \leq 2^\omega$, κ and λ are cardinals $< 2^\omega$, $(f_\alpha \in 2^{\mathbb{N}} : \alpha < \kappa)$ and $(g_\beta \in 2^{\mathbb{N}} : \beta < \lambda)$ are s.t. the Turing ideal generated by $\mathrm{SSy}(M)$ and g_β's does not contain any of f_α. Under what conditions and assuming what forcing axioms, can we find N s.t. $M \prec N$, $\{g_\beta : \beta < \lambda\} \subseteq \mathrm{SSy}(N)$ and $\{f_\alpha : \alpha < \kappa\} \cap \mathrm{SSy}(N) = \emptyset$?

Thank you for your attention.