Failures of Choice in the Blurry HOD Hierarchy

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Blurry HOD

- A set x is *OD* if it is unique such that $\varphi(x, \gamma)$.
- A set x is $<\kappa$-*OD* if it is one of less than κ-many such that $\varphi(x, \gamma)$.

Definition (Fuchs)

A set x is $<\boldsymbol{\kappa}$ **blurrily ordinal definable**, or $<\boldsymbol{\kappa}$-***OD***, if it belongs to an *OD* set of size $< \kappa$.
A set x is $<\boldsymbol{\kappa}$ **hereditarily blurrily ordinal definable**, or $<\boldsymbol{\kappa}$-***HOD***, if $\mathrm{tc}(\{x\}) \subseteq <\kappa\text{-}OD$.

- E.g. $HOD = <2\text{-}HOD$
- Precursors:

  Ordinal algebraic: $<\omega$-*OD* (Hamkins, Leahy)
  Nontypical: $<\omega_1$-*OD* (Tzouvaras)

Basic facts about blurry HOD

Theorem (Fuchs)

(I) *$<\kappa$-HOD is an inner ZF model, which may or may not satisfy AC.*

(II) *Weakly increasing: $<\kappa$-HOD $\subseteq$ $<\lambda$-HOD when $\kappa < \lambda$.*

(III) *$V = \bigcup_{\kappa \in Card} {<}\kappa$-HOD, assuming AC in V.*

Theorem (Blurry Choice; Fuchs)

If $C \in {<}\kappa$-HOD is a set of nonempty sets, there is a function $f \in {<}\kappa$-HOD with domain C that picks, for every $c \in C$, a nonempty subset $f(c) \subseteq c$ of V-cardinality $< \kappa$.

Closeness between levels

The two properties express closeness between two models, introduced by Hamkins.

Theorem (Fuchs)

Let $2 \leq \kappa \leq \lambda$ be infinite cardinals (so $<\kappa$-HOD $\subseteq$ $<\lambda$-HOD).

(I) *$<\kappa$-HOD satisfies the* **external λ-cover property** *in $<\lambda$-HOD: for every $a \in {<\lambda}$-HOD with $a \subseteq {<\kappa}$-HOD and $|a|^V < \lambda$, there is a $c \in {<\kappa}$-HOD with $a \subseteq c$ and $|c|^V < \lambda$.*

(II) *$<\kappa$-HOD satisfies the* **external λ-approximation property** *in $<\lambda$-HOD: if $a \in {<\lambda}$-HOD with $a \subseteq {<\kappa}$-HOD is such that $a \cap c \in {<\kappa}$-HOD for every $c \in {<\kappa}$-HOD with $|c|^V < \lambda$, then $a \in {<\kappa}$-HOD.*

Proof.

(I) If A is a $<\lambda$-blurry ordinal definition of a, let $c := \bigcup A$.

□

Characterizing choiceful levels

Theorem (Fuchs)

The following are equivalent.

(I) $<\kappa$-*HOD* $\models$ *AC*.

(II) $<\kappa$-*HOD is a set forcing extension of HOD by a* κ-*c.c. forcing notion.*

Lemma (Bukovský)

(ZFC) Let M be a transitive inner model of ZFC, κ *an infinite cardinal. TFAE:*

(I) *V is a forcing extension of M by a* κ-*c.c. forcing.*

(II) *Whenever* $f : \alpha \to \beta$ *in V, there is a function* $g : \alpha \to \mathcal{P}(\beta)$ *in M such that* $f(\xi) \in g(\xi)$ *and* $|g(\xi)|^M < \kappa$*, for every* $\xi < \alpha$*.*

Leaps

- Leaps are stages of the hierarchy in which something new appears.

Definition

A cardinal $\lambda > 2$ is a **leap** if $<\delta\text{-}HOD \subsetneq <\lambda\text{-}HOD$ for every $\delta < \lambda$.

- Possible structure of leaps has been studied by Fuchs.

Definition

A leap λ is an ***AC*-leap** if $<\lambda\text{-}HOD \models AC$.
A leap λ is a **non-*AC*-leap** if $<\lambda\text{-}HOD \not\models AC$.

General goal: better understand the possible AC/non-AC patterns in the leaps.

Summary of known choice possibilities

Two *ZFC* facts:

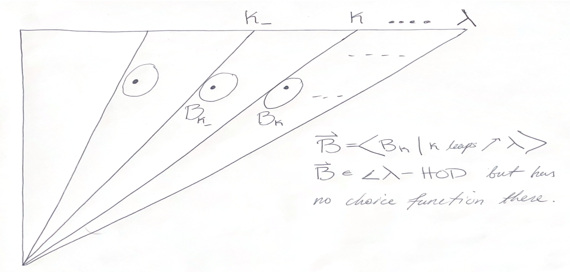
Theorem (Hamkins, Leahy)

$<\omega$-*HOD* $\models AC$. *In fact,* $<\omega$-*HOD* $=$ *HOD*.

$\Longrightarrow$ the first leap is uncountable (if existent).

Theorem (Fuchs)

$<\lambda$-*HOD* $\not\models AC$ *for* λ *a limit of leaps; i.e., a limit of leaps is a non-AC-leap.*



Summary of known leap-choice possibilities

ZFC facts:

- $<\omega$-*HOD* $\models$ *AC* (in fact, $<\omega$-*HOD* $=$ *HOD*; Hamkins, Leahy)
- $<\lambda$-*HOD* $\not\models$ *AC* for λ a limit of leaps (Fuchs)

Consistent with *ZFC* that

- $<\omega_1$-*HOD* $\not\models$ *AC* (Kanovei, Lyubetsky)
- $<\kappa^+$-*HOD* $\not\models$ *AC* for κ inaccessible
- $<\kappa^{++}$-*HOD* $\not\models$ *AC* for $\kappa^{<\kappa} = \kappa$

and all of these are leaps, i.e. non-AC-leaps, and the *least* leaps.

Where we're going

Consistent with *ZFC* that:

- $<\omega_1$-*HOD* $\not\models$ *AC* (Kanovei, Lyubetsky)
- $<\kappa^+$-*HOD* $\not\models$ *AC* for κ inaccessible
- $<\kappa^{++}$-*HOD* $\not\models$ *AC* for κ with $\kappa^{<\kappa} = \kappa$.

Overview:

- All involve a forcing constructed with a $\diamondsuit$-sequence that has a "unique generics" property.
- A product of the forcing is used to add a ρ-sequence of subsets of ρ, the *set* W of which is in $<\rho^+$-*HOD* but not well-orderable there ($\rho = \omega, \kappa, \kappa^+$).
- Will make use of models like $L(W)$ where that result is well-known.

First successor failure of choice

Theorem (Kanovei, Lyubetsky)

In a certain forcing extension of L, $<\omega_1$-HOD fails to satisfy choice.

- Jensen forcing $\mathbb{J}$: consists of perfect subtrees of $2^{<\omega}$ ordered by $\subseteq$, constructed using $\diamondsuit$.
- Kanovei & Lyubetsky extended two key properties to $\mathbb{J}^{<\omega}$:
  (i) $\mathbb{J}^{<\omega}$ is c.c.c.
  (ii) Unique generics property: $\mathbb{J}^{<\omega}$ adds exactly ω-many $\mathbb{J}$-generic subsets of ω, the slices of the filter.

First successor failure of choice

Theorem (Kanovei, Lyubetsky)

In a certain forcing extension of L, $<\omega_1$-HOD fails to satisfy choice.

- Force with $\mathbb{J}^{<\omega}$ to add an ω-sequence $\vec{a} = \langle a_n \mid n < \omega \rangle$ of subsets of ω.
- In $L[\vec{a}]$, only $\mathbb{J}$-generic subsets of ω over L are the a_n.
- So $W = \{a_n \mid n < \omega\}$ is OD and $W \in {<}\omega_1\text{-}HOD$ in $L[\vec{a}]$.
- This yields $L(W) \subseteq {<}\omega_1\text{-}HOD^{L[\vec{a}]}$.

Their proof that $<\omega_1$-$HOD^{L[\vec{a}]} \not\models AC$

The set $W = \{a_n \mid n < \omega\}$ is precisely the set of $\mathbb{J}$-generic subsets of ω over L.
The point: W is not well-orderable in $<\omega_1$-$HOD^{L[\vec{a}]}$.

Lemma

W is not well-orderable in $L(W)$.

Lemma (Kanovei, Lyubetsky)

If $Z \subseteq {}^{\omega}W$ is a countable OD set in $L[\vec{a}]$, then $Z \subseteq L(W)$.

Consequently: If f was a well-ordering of W in $<\omega_1$-$HOD^{L[\vec{a}]}$, then by definition $f \in Z \subseteq L(W)$.

Their proof that $<\omega_1$-$HOD^{L[\vec{a}]} \not\models AC$

In fact we can show $<\omega_1$-$HOD^{L[\vec{a}]} = L(W)$.

Lemma

If Z is a countable OD set in $L[\vec{a}]$, and every $x \in Z$ has $x \subseteq L(W)$, then $Z \in L(W)$.

Corollary

$<\omega_1$-$HOD^{L[\vec{a}]} = L(W) \not\models AC$.

Proof.

If not, take x $\in$-minimal in $<\omega_1$-$HOD^{L[\vec{a}]} \setminus L(W)$, so $x \subseteq L(W)$. Then x belongs to a Z as in the lemma, so $x \in Z \subseteq L(W)$. □

Lemma (Kanovei, Lyubetsky)

If Z is a countable OD set of reals in $V[a]$, then $Z \in V$ (a is Cohen/Solovay-random/dominating/Sacks).

Second successor failure of choice

Friedman & Gitman constructed a Jensen-like forcing $\mathbb{J}(\kappa)$ using $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ for an inaccessible cardinal consisting of perfect κ-trees ordered by $\subseteq$.
$\mathbb{J}(\kappa)$ has same crucial properties as Jensen forcing for previous arguments:

- $<\kappa$-closed.
- κ^+-c.c.
- Adds a unique generic subset of κ.

Theorem (Friedman, Gitman)

In a generic extension $V[G]$ by $\mathbb{J}(\kappa)^{<\kappa}$, the only $\mathbb{J}$-generic/V subsets of κ are the slices of the generic filter.

Second successor failure of choice

Theorem

In an extension of L by $\mathbb{J}(\kappa)^{<\kappa}$, $<\kappa^+$-HOD fails to satisfy choice.

The proof follows the same format as the case for $\mathbb{J}$ with one minor difference:

- The set of generics W is not well-orderable in $L(\mathbb{C}(\kappa, W))$.
  - $\mathbb{C}(\kappa, W)$ is the forcing of *partial* injections $\kappa \rightarrowtail W$ of size $< \kappa$ ordered by extension.
- $<\kappa^+\text{-}HOD^{L[\vec{a}]} = L(\mathbb{C}(\kappa, W))$.

In the case of $\mathbb{J}(\omega) = \mathbb{J}$, $\mathbb{C}(\omega, W)$ is also used, but there we have $L(\mathbb{C}(\omega, W)) = L(W)$.

Successor non-AC leaps

In both cases of $<\kappa^+$-$HOD \not\models AC$ ($\kappa = \omega$ or κ inaccessible), κ^+ is the least leap. Moreover, GCH still holds in the extension of L.

Lemma (Fuchs)

If κ is a regular cardinal, $\mathbb{P}$ a cone homogenous, $<\kappa$-closed forcing notion, and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then $<\kappa$-$HOD^{V[G]} \subseteq V$.

Corollary

For $\kappa = \omega$ or κ inaccessible, it is consistent that κ^+ is a non-AC-leap and the least leap.

Proof.

In the previous results, $\mathbb{P} = \mathbb{J}(\kappa)^{<\kappa}$ is cone homogenous and $<\kappa$-closed, so in fact $<\kappa$-$HOD^{L[\vec{a}]} = L$, meanwhile $W \in <\kappa^+$-$HOD^{L[\vec{a}]} \setminus L$. □

Distilling the crucial properties of the forcings

In showing the inclusion $L(\mathbb{C}(\kappa, W)) \subseteq {<}\kappa^+\text{-}HOD^{L[\vec{a}]}$, the unique generics property of the forcing was essential.

Definition

A poset $\mathbb{P}$ is **κ-Jensen** if $\mathbb{P}^{<\kappa}$ preserves cardinals, and in an extension by $\mathbb{P}^{<\kappa}$ the only $\mathbb{P}$-generics over the ground are the κ-many slices of the $\mathbb{P}^{<\kappa}$-generic.

Examples

(i) Jensen forcing $\mathbb{J}$ is ω-Jensen.

(ii) Jensen forcing at an inaccessible $\mathbb{J}(\kappa)$ is κ-Jensen.

Lemma

If $\mathbb{P} \in L$ is κ-Jensen, then $L(\mathbb{C}(\kappa, W)) \subseteq {<}\kappa^+\text{-}HOD^{L[\vec{a}]}$, where $\vec{a}$ is $\mathbb{P}^{<\kappa}$-generic/L, and W is the set of these κ-many $\mathbb{P}$-generics.

Distilling the crucial properties of the forcings

In showing the inclusion $<\kappa^+$-$HOD^{L[\vec{a}]} \subseteq L(\mathbb{C}(\kappa, W))$, a different crucial property of the Jensen forcings is essential.
Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$.

Definition

A poset $\mathbb{P}$ is **κ-Kanovei** if the following hold.

(i) $\mathbb{P}^{<\kappa}$ does not add a new bounded subset of κ.

(ii) Let G be $\mathbb{P}^{<\kappa}$-generic/V. Whenever V_θ is sufficiently correct and $\pi : \overline{V} \to V_\theta$ is an elementary embedding with $\overline{V}$ transitive and of size κ, and $(\kappa + 1) \cup \{\mathbb{P}^{<\kappa}\} \subseteq \mathrm{ran}(\pi)$, then letting $\overline{\mathbb{P}} := \pi^{-1}(\mathbb{P})$ and $\overline{G} := \pi^{-1}{}'' G$, $\overline{G}$ is $\overline{\mathbb{P}}^{<\kappa}$-generic/$\overline{V}$ and for every $\delta < \kappa$, there is in V a $\mathbb{C}(\kappa, \kappa)$-generic/$\overline{V}[\overline{G}]$ function $I : \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow \kappa$ such that $I \restriction \delta = \mathrm{id}$.

Example

$\mathbb{J}(\kappa)$ for $\kappa = \omega$ or κ inaccessible is κ-Kanovei.

Getting $<\kappa^+$-$HOD^{V[G]} \subseteq V(\mathbb{C}(\kappa, W))$ for $\mathbb{P}$ κ-Kanovei

Useful tool: we can view a $\mathbb{P}^{<\kappa}$-generic sequence $\vec{a} = \langle a_\xi \mid \xi < \kappa \rangle$ as a $\mathbb{C}(\kappa, W)$-generic function$/V(\mathbb{C}(\kappa, W))$, and vice versa.

Lemma (Kanovei, Lyubetsky)

If $\vec{a}$ is a $\mathbb{P}^{<\kappa}$-generic sequence over V, then $\vec{a}$ is a $\mathbb{C}(\kappa, W)$-generic function over $V(\mathbb{C}(\kappa, W))$.

Lemma (Kanovei, Lyubetsky)

If $\vec{b}$ is a $\mathbb{C}(\kappa, W)$-generic function over $V(\mathbb{C}(\kappa, W))$, then $\vec{b}$ is a $\mathbb{P}^{<\kappa}$-generic sequence over V.

Getting $<\kappa^+$-$HOD^{V[G]} \subseteq V(\mathbb{C}(\kappa, W))$ for $\mathbb{P}$ κ-Kanovei

Let $\mathbb{P}$ be κ-Kanovei and G be $\mathbb{P}^{<\kappa}$-generic$/V$.

Lemma

If $A \in V[G]$ is definable from parameters in $V(\mathbb{C}(\kappa, W))$ and of size $\leq \kappa$, and every $x \in A$ has $x \subseteq V(\mathbb{C}(\kappa, W))$, then $A \in V(\mathbb{C}(\kappa, W))$. Thus, $<\kappa^+$-$HOD^{V[G]}_{V(\mathbb{C}(\kappa,W))} = V(\mathbb{C}(\kappa, W))$.

Proof.

Fix $p_0 \in \mathbb{C}(\kappa, W)$, extended by $\bigcup G : \kappa \rightarrowtail\!\!\!\!\rightarrow W$, forcing our assumptions over $V(\mathbb{C}(\kappa, W))$:

- $\dot{A} = \{x \mid \varphi(x, \check{v})\}$ is size κ, $v \in V(\mathbb{C}(\kappa, W))$.
- Every $x \in \dot{A}$ has $x \subseteq V(\check{\mathbb{C}}(\kappa, W))$.

Let $\mathbb{C} = \mathbb{C}(\kappa, W)$.

$A \in OD^{V[G]}_{V(\mathbb{C})}$, $|A| \leq \kappa$, $A \subseteq \mathcal{P}(V(\mathbb{C})) \Rightarrow A \in V(\mathbb{C})$

Proof.

$p_0 \in \mathbb{C}$, extended by $\bigcup G : \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow W$, $\mathbb{C}$-forces/$V(\mathbb{C})$ our assumptions on $\dot{A}$.

Sketch:

- Pass to sufficiently correct transitive $\overline{V}$ of size κ, where $\overline{G}$, $\overline{\mathbb{P}}$ are the collapses, $\overline{\mathbb{C}}$ in $\overline{V}[\overline{G}]$, $\overline{\dot{A}}$ in $\overline{V}(\overline{\mathbb{C}})$.
- Use κ-Kanovei-ness to get, in V, $I : \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow \kappa$, $\mathbb{C}(\kappa,\kappa)$-generic over $\overline{V}[\overline{G}]$ with $I \restriction \mathrm{dom}(p_0) = \mathrm{id}$.
- Permute coordinates of $\overline{G}$ by I to get $\overline{H}$ mutually $\overline{\mathbb{P}}^{<\kappa}$-generic/$\overline{V}$ with $\overline{G}$; yet $V[G] = V[H]$.
  $\Rightarrow \bigcup \overline{G}$ and $\bigcup \overline{H}$ mutually $\overline{\mathbb{C}}$-generic/$\overline{V}(\overline{\mathbb{C}})$.
- Show $\overline{\dot{A}}^{\overline{G}} = \overline{\dot{A}}^{\overline{H}} (=: \overline{A})$.
- Then $\overline{A} \in \overline{V}(\overline{\mathbb{C}})[\overline{G}] \cap \overline{V}(\overline{\mathbb{C}})[\overline{H}]$, so $\overline{A} \in \overline{V}(\overline{\mathbb{C}})$, which will get $A \in V(\mathbb{C})$.

□

Summary

Morally:

- κ-Kanovei $\Rightarrow$ $<\!\kappa^+$-$HOD^{V[\vec{a}]} \subseteq V(\mathbb{C}(\kappa, W))$.

- κ-Jensen $\Rightarrow$ $L(\mathbb{C}(\kappa, W)) \subseteq\, <\!\kappa^+$-$HOD^{L[\vec{a}]}$.

- κ-Jensen + κ-Kanovei $\Rightarrow$ $<\!\kappa^+$-$HOD^{L[\vec{a}]} = L(\mathbb{C}(\kappa, W))$ fails to satisfy choice.

Removing the inaccessible: free Suslin trees

- Work of Krueger shows a similar "unique generics" phenomenon with free Suslin trees.

Definition

A Suslin tree T is **free** if, for any finitely many nodes $s_0, \ldots, s_{n-1}$ on the same level of T, the tree $T_{s_0} \bigotimes \cdots \bigotimes T_{s_{n-1}}$ is Suslin. Equivalently, $T_{s_0} \times \cdots \times T_{s_{n-1}}$ is c.c.c.

- $T_s = \{v \in T \mid s \leq_T v\}$
- The tree product $\bigotimes$ only takes tuples of nodes from the same level.
- Freeness is a high form of rigidity, and free Suslin trees exist under the assumption of $\diamondsuit$.

Removing the inaccessible: free Suslin trees

- Let T be a *free, normal* Suslin tree and let $\mathbb{P}$ be the λ-fold product of T with countable support.
- Forcing with $\mathbb{P}$ adds branches b_i through T for $i < \lambda$, subsets of ω_1.

Theorem (Krueger)

$\mathbb{P}$ is countably distributive and $(2^\omega)^+$-Knaster.

Corollary

Under $\diamondsuit$, $\mathbb{P}$ preserves cardinals.

Theorem (Krueger)

$\mathbb{P}$ adds exactly the branches b_i for $i < \lambda$.

Theorem

Under $\diamondsuit$, T is ω_1-Jensen.

Extending to free κ^+-Suslin trees

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$.

- Let T be a *free, normal,* $<\kappa$*-closed* κ^+-Suslin tree — such exist under $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$.
- Let $\mathbb{P}$ be the λ-fold product of T with κ-support.

Theorem

$\mathbb{P}$ is $<\kappa^+$-distributive and $(2^\kappa)^+$-Knaster.

Corollary

Under $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$, $\mathbb{P}$ preserves cardinals.

Theorem

$\mathbb{P}$ adds exactly the branches b_i for $i < \lambda$.

Theorem

Under $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$, T is κ^+-Jensen.

Extending to free κ^+-Suslin trees

Theorem

T is κ^+-Jensen; so $L(\mathbb{C}(\kappa^+, B)) \subseteq {<}\kappa^{++}$-HOD$^{L[\vec{b}]}$.

Theorem

T is κ^+-Kanovei; so ${<}\kappa^{++}$-HOD$^{L[\vec{b}]} \subseteq L(\mathbb{C}(\kappa^+, B))$.

Theorem

In $L[\vec{b}]$, ${<}\kappa^{++}$-HOD $= L(\mathbb{C}(\kappa^+, B))$ fails to satisfy choice.

Theorem

In $L[\vec{b}]$, κ^{++} is a non-AC-leap, and the least leap.

Proof.

$B = \{b_i \mid i < \kappa^+\}$ is in ${<}\kappa^{++}$-*HOD* but b_0 is not in ${<}\kappa^+$-*HOD*. □

$\kappa^{<\kappa} = \kappa$, T is a normal, free, $<\kappa$-closed κ^+-Suslin tree

Theorem

$\mathbb{P} = \prod_{\lambda}^{\leq\kappa} T$ adds exactly the branches b_i for $i < \lambda$.

Definition (Pseudo-forcing)

Let θ be a regular cardinal, $p \in \mathbb{P}$. Let $p, \mathbb{P} \in X \prec H_\theta$, where $\kappa + 1 \subseteq X$ and X has cardinality κ. Define $\mathbb{Q}_{X,p}$ as follows. Let $\delta := \kappa^+ \cap X$. Conditions are of the form $\langle r, q, I \rangle$ where

(i) $r \in \mathbb{P} \cap X$,

(ii) $r \leq_{\mathbb{P}} p$,

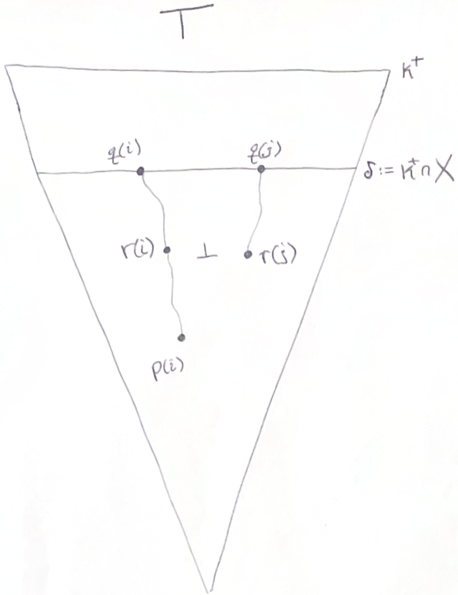
(iii) $I \in X$ is a subset of $\operatorname{dom}(r)$ of size $< \kappa$,

(iv) for $i \neq j$ in I, $r(i)$ and $r(j)$ are incomparable in T,

(v) $q : I \to T(\delta)$,

(vi) for $i \in I$, $r(i) <_T q(i)$.

Pseudo forcing



$$\langle r, q, I \rangle \in \mathbb{Q}_{X,P}$$

$$q : I \to T(\delta)$$

$$I \subseteq \operatorname{dom} r$$

$$|I| < \kappa$$

$$r \in \mathbb{P} = \prod_{\lambda}^{\leq \kappa} T$$

$$r : \operatorname{dom} r \to T$$

$$|\operatorname{dom} r| \leq \kappa$$

$$\operatorname{dom} r \subseteq \lambda$$

$\mathbb{P} = \prod_{\lambda}^{\leq\kappa} T$ adds exactly the branches b_i for $i < \lambda$

Lemma (Existence of pseudo-generics)

If $^{<\kappa}X \subseteq X$, then given a collection $\mathcal{D}$ of κ-many dense subsets of $\mathbb{Q}_{X,p}$, there is a $\mathcal{D}$-generic filter G.

Lemma

The following are dense in $\mathbb{Q} = \mathbb{Q}_{X,p}$.

(1) $\Delta_i^I = \{\langle r, q, I\rangle \in \mathbb{Q} \mid i \in I\}$ *for every* $i \in \lambda \cap X$.

(2) $\Delta_\xi^r = \{\langle r, q, I\rangle \in \mathbb{Q} \mid \forall i \in I\ |r(i)| \geq \xi\}$ *for every* $\xi < \delta$ $(= \kappa^+ \cap X)$

(3) *Assuming* $\lambda \geq \kappa^+$, $\Delta_z = \{\langle r, q, I\rangle \in \mathbb{Q} \mid z \in ran(q)\}$ *for every* $z \in T(\delta)$.

If $G \subseteq \mathbb{Q}$ is a filter meeting all of these (κ-many) dense sets, then for $\hat{q} = \bigcup\{q \mid \exists, r, I\ \langle r, q, I\rangle \in G\}$, $\hat{q} \in \mathbb{P}$, $dom(\hat{q}) = \lambda \cap X$, $\hat{q} \leq_{\mathbb{P}} p$, $\hat{q} \leq_{\mathbb{P}} r$ *for* $\langle r, q, I\rangle \in G$, *and* $ran(\hat{q}) = T(\delta)$.

$\mathbb{P} = \prod_{\lambda}^{\leq\kappa} T$ adds exactly the branches b_i for $i < \lambda$

Lemma

If $D \subseteq \mathbb{P}$ is dense below p, where $p, D \in X$, then $D^{\mathbb{Q}} = \{\langle r, q, I\rangle \in \mathbb{Q} \mid r \in D\}$ is dense in $\mathbb{Q}$.

Theorem

$\mathbb{P}$ adds exactly the branches b_i for $i < \lambda$.

Proof.

- If not, let $p \in \mathbb{P}$ force that $\dot{b}$ is cofinal branch through T, but $\dot{b} \neq \dot{b}_i$ for all $i < \lambda$.
- Throw everything in to a κ-size $X \prec H_\theta$ from before, ${}^{<\kappa}X \subseteq X$.
- For $i \in \lambda \cap X$, let D_i be open dense below p set of $r \leq_{\mathbb{P}} p$ that force $\dot{b} \restriction \xi \neq \dot{b}_i \restriction \xi$ for some $\xi < \kappa^+$.

□

$\mathbb{P} = \prod_{\lambda}^{\leq\kappa} T$ adds exactly the branches b_i for $i < \lambda$

Theorem

$\mathbb{P}$ adds exactly the branches b_i for $i < \lambda$.

Proof.

- Let $p \in \mathbb{P}$ force that $\dot{b}$ is cofinal branch through T, but $\dot{b} \neq \dot{b}_i$ for all $i < \lambda$.
- Throw parameters into κ-size $\mathrm{X} \prec H_\theta$ from before, ${}^{<\kappa}X \subseteq X$.
- For $i \in \lambda \cap X$, let D_i be open dense below p set of $r \leq_{\mathbb{P}} p$ that force $\dot{b} \restriction \xi \neq \dot{b}_i \restriction \xi$ for some $\xi < \kappa^+$.
- Let G be psuedo-generic, meeting all the κ-many dense sets from the previous lemma plus all $D_i^{\mathbb{Q}}$.
- For every $i \in \lambda \cap X$, $\hat{q}$ $\mathbb{P}$-forces that $\dot{b} \restriction \delta \neq \dot{b}_i \restriction \delta$.
- $\mathrm{ran}(\hat{q}) = T(\delta)$, so $\hat{q} \leq_{\mathbb{P}} p$ forces $\dot{b}$ is not a cofinal branch through T, contradiction.

□

Summary

It is consistent the following are non-AC-leaps, i.e., something new appears in their level of the blurry HOD hierarchy and their level fails to satisfy choice. Moreover, in each case, they are the least leap.

- ω_1 (Jensen forcing) (Kanovei)
- κ^+, for κ inaccessible (Jensen forcing at an inaccessible)
- κ^{++}, with $\kappa^{<\kappa} = \kappa$ (free κ^+-Suslin tree)

Questions.
Can we arrange for multiple non-AC-leaps simultaneously?
What are some other κ-Kanovei or κ-Jensen posets?

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