On Untranscendable Order Types

Thilo Weinert

CUNY Set Theory Seminar, Friday, 27th March 2026

Joint work with Garrett Ervin and Alberto Marcone

1868	is born in Breslau (today Wrocław).
1887–1891	studies mathematics and astronomy.
1901	becomes a professor in Leipzig.
1901	lectures on set theory.
1908	'Grundzüge einer Theorie der geordneten Mengen'.
1913	becomes a professor in Greifswald.
1914	'Grundzüge der Mengenlehre'.
1921	becomes a professor in Bonn.
1935	becomes emeritus.
1939	tries to attain a research fellowship in the USA.
1942	takes his own life after facing deportation.

Question

What is a number?

Definition

A *linear ordering* (or a *chain*) is a set with an irreflexive, asymmetric, transitive binary relation on it.

We are mainly interested in the *order type* of an ordering.

Definition

$$\omega = \mathsf{otp}(\mathbb{N}),$$
$$\eta = \mathsf{otp}(\mathbb{Q}),$$
$$\lambda = \mathsf{otp}(\mathbb{R}).$$

Definition

For any order type φ, its reverse is denoted by φ^*.

Suppose that ρ and φ are order types. Let $(X, <_\rho)$ be an order of type ρ and $(Y, <_\varphi)$ be one of type φ.

Definition

$\rho + \varphi$ is the order type of $X \times \{0\} \cup Y \times \{1\}$ equipped with the order which puts $x < y$ if

- $x \in X$ and $y \in Y$ or
- $x <_\rho y$ or
- $x <_\varphi y$.

Definition

$\rho\varphi$ is the order type of $X \times Y$ ordered antilexicographically.

Definition

We write $\rho \leqslant \varphi$ to express that every ordering of type ρ can be order-preservingly embedded into one of type φ. We write $\rho < \varphi$ to abbreviate ($\rho \leqslant \varphi$ but $\varphi \not\leqslant \rho$).

Definition (Hausdorff, 1908)

An order type φ is called *scattered* if $\eta \not\leqslant \varphi$.

Theorem (Hausdorff)

The class of scattered orders is the smallest nonempty class of order types closed under both under reversals and well-ordered sums.

Observation

Linear order-types are not linearly ordered by embeddability, consider ω and ω^*.

Definition

Two order-types ρ and φ such that both $\rho \leqslant \varphi$ and $\varphi \leqslant \rho$ are called *equimorphic*, we write $\rho \simeq \varphi$.

Example

Equimorphism between order types fails to imply equality, consider $(\omega^* + \omega)\omega$ and $(\omega^* + \omega + 1)\omega$.

Definition

An order-type ρ is called *decomposable* if there are order-types $\varphi, \psi < \rho$ such that $\varphi + \psi = \rho$ and *indecomposable* otherwise.

Definition

An order-type ρ is called *strictly indecomposable to the left* if whenever $\varphi + \psi = \rho$, then $\varphi = \rho$ and $\psi < \rho$.

Definition

An order-type ρ is called *strictly indecomposable to the right* if whenever $\varphi + \psi = \rho$, then $\varphi < \rho$ and $\psi = \rho$.

Observation

Any order type φ is strictly indecomposable to the left if and only if φ^* is strictly indecomposable to the right.

Definition

An order-type φ is called *strongly indecomposable* if for every ordering $(X, <_\varphi)$ of type φ and every $Y \subset X$, the ordering $(X, <_\varphi)$ can be embedded order-preservingly into $(Y, <_\varphi)$ or $(X \setminus Y, <_\varphi)$.

Observation

Every strongly indecomposable order-type is indecomposable.

Fact

An ordinal number is indecomposable if and only if it is strongly indecomposable.

Observation

If $\varphi^2 \leqslant \varphi$, then φ is strongly indecomposable.

Example

ω is strictly indecomposable to the right and strongly indecomposable.

Example

$(\omega^* + \omega)\omega$ is strictly indecomposable to the right but not strongly indecomposable.

Examples

0, 1, and η are strongly indecomposable but not strictly indecomposable to either side.

Example (Sierpiński, using AC, 1950)

λ is indecomposable, but neither strictly to the left, nor strictly to the right, nor strongly so (the last according to AC).

Theorem (Jullien, 1968)

Every scattered indecomposable order-type is either strictly indecomposable to the left or to the right.

Observation

If $\varphi 2 \leqslant \varphi$ then $\varphi = 0$ or $\eta \leqslant \varphi$.

Theorem (Hagendorf, 1977)

Every indecomposable order-type $\varphi > 0$ such that $\varphi 2 \not\leqslant \varphi$ is strictly indecomposable to the left or to the right.

Corollary (Reformulation)

Every indecomposable order-type φ such that both $1 + \varphi \leqslant \varphi$ and $\varphi + 1 \leqslant \varphi$ also satisfies $\varphi 2 \leqslant \varphi$.

Folklore

The infinite additively indecomposable ordinals are exactly those of the form ω^{α} for some positive ordinal α.

Definition (Hessenberg, 1906)

An ordinal is a δ-number if it is an indecomposable ω^{β} whose exponent β is indecomposable itself.

Definition (Jacobsthal, 1909)

An ordinal α is said to be multiplicatively principal if $0 < \beta < \alpha$ implies $\beta\alpha = \alpha$.

Definition (Bachmann, 1967)

An ordinal is α called *multiplicatively decomposable* if there are ordinals $\beta, \gamma < \alpha$ such that $\beta\gamma = \alpha$, otherwise it is called multiplicatively *indecomposable*.

Definition

An order type φ is *transcendable*, if there are order types $\psi, \tau < \varphi$ such that $\varphi \leqslant \psi\tau$ and *untranscendable* otherwise.

Definition

An order type φ is *s-untranscendable* if whenever ψ and τ are order types such that $\varphi \leqslant \psi\tau$, then $\varphi \leqslant \psi$ or $\varphi \leqslant \tau$.

Proposition

Suppose α is an ordinal. Each of the following statements implies the next one and if $\alpha > 2$, then they are all equivalent

1. $\alpha = \omega^{\omega^{\beta}}$ *for some ordinal* β,

2. α *is a* δ-*number*,

3. α *is untranscendable.*

Proposition

An ordinal is s-untranscendable if and only if it is untranscendable but no singular limit.

Example

$$\omega^{\omega} \leqslant \left(\sum_{n<\omega}^{*} \omega^n \right) \omega.$$

Proposition

Suppose ρ is any order type. Then both the type

$$\tau = \sum_{n \in \omega} \rho^n = 1 + \rho + \rho^2 + \cdots$$

and the type

$$\tau' = \sum_{n \in \omega}^{*} \rho^n = \cdots + \rho^2 + \rho + 1$$

are untranscendable.

Corollary

For every order type ρ, there is an untranscendable order type $\tau > \rho$.

Theorem (Siksek, 2015)

Every natural number besides
15, 22, 23, 50, 114, 167, 175, 186, 212, 238, 239, 303, 364, 420, 428, 454
is the sum of at most seven positive cubes.

Moral

Never let exceptions keep you from proving a theorem!

Proposition

Suppose φ, ψ, τ, and ρ are order types and $\varphi + \psi \leqslant \tau + \rho$. Then either $\varphi \leqslant \tau$ or $1 + \psi \leqslant \rho$. Symmetrically, either $\varphi + 1 \leqslant \tau$ or $\psi \leqslant \rho$.

Corollary

Suppose that φ is a type such that $\varphi 2 \leqslant \varphi$. Then φ is indecomposable.

Lemma

If $\psi \neq 0$, then $\psi + \tau \leqslant \psi(1 + \tau)$ and
if $\tau \neq 0$, then $\psi + \tau \leqslant \tau(\psi + 1)$.

Theorem

2 is the unique decomposable untranscendable order type.

Proof.

Assume that $\psi + \tau$ is untranscendable but decomposable, witnessed by ψ
and τ, so $\psi + \tau \not\leqslant \psi$ and $\psi + \tau \not\leqslant \tau$. The latter implies that $1 \leqslant \psi$. As
$\psi + \tau$ is untranscendable, the Lemma implies that $\psi + \tau \leqslant 1 + \tau$. Were
$2 \leqslant \psi$ true, then we would have $2 + \tau \leqslant 1 + \tau$. As $2 \not\leqslant 1$ we have
$1 + \tau \leqslant \tau$ by the earlier Proposition. We obtain $\psi + \tau \leqslant \tau$ by transitivity,
a contradiction. Therefore, $\psi = 1$. Similarly one can show that $\tau = 1$.
So $\psi + \tau = 2$. □

Definition

A suborder I of a linear order X is an *interval* of X if whenever $x < y < z$ are points in X and $x, z \in I$ then $y \in I$. We say that an interval is *non-degenerate* if it contains at least two points. An interval is *open* if it has neither a left nor right endpoint, and *closed* if it has both a right and left endpoint.

Definition

A linear order X is *homogeneous* if whenever $I \subseteq X$ is a non-degenerate interval in X, then $X \leqslant I$.

An order type φ is *homogeneous* if some (equivalently every) linear order X of type φ is homogeneous.

Examples

0, 1, and 2 are homogeneous.

Examples

η, $1 + \eta$, $\eta + 1$, and $1 + \eta + 1$ are homogeneous.

Examples

λ, $1 + \lambda$, $\lambda + 1$, and $1 + \lambda + 1$ are homogeneous.

Example

Homogeneity is no equimorphism invariant—consider η and 2η.

Proposition

All homogeneous types are s-untranscendable.

Corollary

λ is s-untranscendable.

Lemma (AC)

Suppose φ is an s-untranscendable type such that $\varphi 2 \leqslant \varphi$. Then there is a homogeneous type φ' such that $\varphi' \simeq \varphi$.

Theorem (AC)

Suppose ρ, τ, φ and ψ are order types with ρ, φ not both 0. If $\rho\tau \leqslant \varphi\psi$ then $\tau \leqslant \psi$ or both $1 + \rho \leqslant \varphi$ and $\rho + 1 \leqslant \varphi$.

Corollary (AC)

If ρ, τ, φ and ψ are such that $\rho\tau \leqslant \varphi\psi$ then $\rho \leqslant \varphi$ or $\tau \leqslant \psi$.

Corollary (AC)

If φ is such that $\varphi^2 \leqslant \varphi$, then φ is s-untranscendable.

Theorem (AC)

If φ is s-untranscendable and both $\varphi 2 \leqslant \varphi$ and $2\varphi \leqslant \varphi$, then $\varphi^2 \leqslant \varphi$.

Examples

No two of the assumptions suffice to prove the theorem, as shown by the order types $\omega\lambda$, ω, and λ.

Definition (Higman and Erdős–Rado, 1952)

An ordering is a *well-quasi-ordering* if it neither has infinite descending chains nor infinite antichains.

Conjecture (Fraïsse)

The countable linear orderings are well-quasi-ordered by embeddability.

Theorem (Laver)

The class of σ-scattered linear orderings is well-quasi-ordered by embeddability.

Theorem

Every σ-scattered order type is a finite sum of indecomposable σ-scattered order types.

Definition (Galvin)

An order $(X, <)$ (and its corresponding type) is called σ-scattered if there is a decomposition $X = \bigcup_{n < \omega} X_n$ such that for every natural number n the induced suborders $(X_n, <)$ is scattered.

Definition

An *Aronszajn type* is an uncountable order type φ such that

- $\omega_1 \not\leqslant \varphi$,
- $\omega_1^* \not\leqslant \varphi$, and
- all order types ψ such that both $\psi \leqslant \varphi$ and $\psi \leqslant \lambda$ are countable.

Theorem (Aronszajn, using AC)

There is an Aronszajn type.

Definition (Countryman)

A *Countryman line* is an uncountable linear ordering $(X, <)$ such that $X \times X$ is the union of countably many chains in the sense of $(a, b) \leqslant (c, d)$ iff $a \leqslant c \wedge b \leqslant d$.

Conjecture (Countryman)

There are no Countryman lines.

Theorem (Baumgartner, 1971)

Consistently, any two $\aleph_1$-dense sets of reals are order-isomorphic.

Theorem (Shelah, using AC, 1975)

There is a Countryman line.

Conjecture (Shelah)

PFA *implies that every Aronszajn type contains a Countryman type.*

Definition

For classes $\mathcal{B}, \mathcal{C}$ of order types, $\mathcal{B} \subset \mathcal{C}$ is a basis for $\mathcal{C}$ if there is a $\varphi \in \mathcal{B}$ such that $\varphi \leqslant \rho$ for every $\rho \in \mathcal{C}$.

Example

$\{\omega, \omega^*\}$ is a basis for the infinite linear orders.

Theorem (Moore, 2006)

PFA *implies that for any Countryman type* γ, *the set* $\{\gamma, \gamma^*\}$ *is a basis for the Aronszajn types.*

Corollary

PFA *implies that there is a five-element basis for the uncountable linear orders.*

Theorem (Moore, 2009)

PFA *implies that there is a universal Aronszajn type, pick one and call it* ν.

Definition (PFA)

An Aronszajn type φ is called *fragmented* if $\nu \not\leq \varphi$.

Theorem (Martinez-Ranero, 2011)

PFA *implies that the class of Aronszajn types is well-quasi-ordered by the embeddability relation.*

Definition

Given points $x \leq y$ in X we write $[x, y]$ for the interval $\{z \in X : x \leq z \leq y\}$. For an arbitrary pair of points $x, y \in X$, we write $[\{x, y\}]$ to denote the interval $[x, y]$ when $x \leq y$ and $[y, x]$ when $y \leq x$.

Definition

Given a linear order X, define a binary relation F on X by the rule xFy if and only if $[\{x, y\}]$ is finite. For a given $x \in X$, we refer to the condensation class $[x]_F$ as the *F-class* of x.

Definition

Suppose φ is an order type. A *φ-shuffle* (or *shuffle over φ*) is an ordered sum of the form $\sum_{x \in \varphi} \psi_x$, where for each $x \in \varphi$ the set $\{y \in \varphi : \psi_x \leqslant \psi_y\}$ is dense in φ.

Definition

Suppose κ is an infinite regular cardinal. A *regular unbounded κ-sum* is an ordered sum of the form $\sum_{\alpha \in \kappa} \varphi_\alpha$, where for each $\alpha \in \kappa$ the set $\{\beta \in \kappa : \varphi_\alpha \leqslant \varphi_\beta\}$ is unbounded (to the right) in κ.
A *regular unbounded κ^*-sum* is an ordered sum $\sum_{\alpha \in \kappa^*} \varphi_\alpha$, where for each $\alpha \in \kappa^*$, $\{\beta \in \kappa^* : \varphi_\alpha \leqslant \varphi_\beta\}$ is unbounded (to the left) in κ^*.
A *regular unbounded sum* is a regular unbounded κ-sum or regular unbounded κ^*-sum for some infinite regular cardinal κ.

Theorem

Suppose that $\mathcal{F}$ is a class of order types such that for every $\rho \in \mathcal{F}$, either $\rho \simeq \rho^2$ or one of ρ, ρ^* is an infinite regular cardinal.
Let $\mathcal{T}$ denote the class of order types obtained by closing $\{0, 1\}$ under ρ-shuffles (when $\rho \equiv \rho^2$) and regular unbounded ρ-sums (when one of ρ, ρ^* is an infinite regular cardinal), for all $\rho \in \mathcal{F}$.
Then for every $\varphi \in \mathcal{T}$, either $\varphi = 1$ or φ is equimorphic to a type $\tilde{\varphi}$ without finite $\mathcal{F}$-classes.

Theorem

Let φ be an untranscendable order type that is equimorphic to one with only finitely many finite F-classes. Then φ is strongly indecomposable or $\varphi = 2$.

Example

Having finitely many finite F-classes is not equimorphism invariant, consider η and $(\omega^* + \omega)\eta$.

Theorem

Every σ-scattered linear order is equimorphic to a linear order with only finitely many finite F-classes.

Theorem

Every untranscendable σ-scattered linear order different from 2 is strongly indecomposable.

Definition

Let $\mathcal{H}$ denote the class of linear order types obtained by closing $\{0, 1\}$ under φ-regular unbounded sums, for $\varphi \in \{\omega, \omega^*\}$, and under φ-shuffles, for $\varphi \in \{\eta, \gamma, \gamma^*\}$.

Definition

$\mathcal{C}$ denotes the class consisting all fragmented Aronszajn types along with all countable types.

Theorem (Barbosa, 2023)

PFA *implies that every type* $\varphi \in \mathcal{C}$ *can be written as a finite sum of members of* $\mathcal{H}$.

Theorem (Barbosa, 2023)

PFA *implies that the indecomposable members of* $\mathcal{C}$ *are exactly the types equimorphic to an element of* $\mathcal{H}$.

Theorem

PFA *implies that every member of* $\mathcal{H}$ *except* 1 *is equimorphic to a type without finite F-classes.*

Corollary

PFA *implies that every member of* $\mathcal{C}$ *is equimorphic to a linear order with only finitely many finite F-classes.*

Corollary

PFA *implies that every untranscendable member of* $\mathcal{C}$ *except* 2 *is strongly indecomposable.*

Definition

A *Bernstein set* is a set of reals that intersects every perfect set but does not contain any perfect set.

Folklore (ZF only)

λ is strongly indecomposable if and only if there is no Bernstein set.

Folklore (ZF only)

No Bernstein set has the property of Baire.

Theorem (ZF only, Shelah, 1980)

If ZF is consistent, then there is a model of ZF in which every set of reals has the property of Baire.

Corollary

$ZF + $ 'λ is strongly indecomposable.' is consistent.

Problem

Is there an untranscendable order type φ such that $\varphi \simeq \varphi 2 \simeq 2\varphi$ but $\varphi \not\simeq \varphi^2$?

Conjecture (The Trichotomy Conjecture (possibly assuming PFA))

For every untranscendable linear order φ at least one of the following three statements applies:

- *φ is strongly indecomposable,*
- *φ contains an uncountable real type,*
- *$\varphi = 2$.*

Question

Does ZFC prove that all untranscendable Aronszajn lines are strongly indecomposable?

Question

Is it provable with ZF that if $\rho\tau \leqslant \varphi\psi$, then necessarily $\rho \leqslant \varphi$ or $\tau \leqslant \psi$?

Definition

BE (**B**inary **E**xceptionalism) states that 2 is the only untranscendable linear ordering failing to be strongly indecomposable.

Definition

O states that every set can be ordered linearly.

Definition

DC denotes the principle of dependent choices.

Definition

NBS states that there is no Bernstein set.

Question

Is ZF + BE *consistent?*

Question

Is ZF + BE + O *consistent?*

Question

Is ZF + DC + BE + O *consistent?*

Folklore

$ZF + AD$ *proves that* $\mathbb{R}/\mathbb{Q}$ *cannot be ordered linearly.*

Personal Communication (Glazer, Karagila)

It is open whether $ZF + O + NBS$ *is consistent.*

Thank you for your attention!

Heinz Bachmann. Transfinite Zahlen. 2. neubearb. Aufl. Berlin-Heidelberg-New York: Springer-Verlag. VIII, 228 p. (1967)., 1967.

Keegan Dasilva Barbosa. Decomposing Aronszajn lines. *J. Math. Log.*, 23(1):Paper No. 2250017, 20, 2023. ISSN 0219-0613,1793-6691. doi:10.1142/S0219061322500179.

James Earl Baumgartner. All $\aleph_1$-dense sets of reals can be isomorphic. *Fund. Math.*, 79(2):101–106, 1973. ISSN 0016-2736. doi:10.4064/fm-79-2-101-106. URL https://doi.org/10.4064/fm-79-2-101-106.

Paul Bernays. A system of axiomatic set theory. III: Infinity and enumerability. Analysis. *J. Symb. Log.*, 7:65–89, 1942. ISSN 0022-4812. doi:10.2307/2266303. URL doc.rero.ch/record/290936/files/S0022481200064185.pdf.

Garrett Ervin, Alberto Marcone, and Thilo Weinert. Untranscendable order types. Submitted, 2026. URL https://arxiv.org/abs/2602.24285.

Roland Fraïssé. Sur la comparaison des types d'ordres. *C. R. Acad. Sci. Paris*, 226:1330–1331, 1948. ISSN 0001-4036.

Jean Guillaume Hagendorf. *Extensions de chaînes.* PhD thesis, Univ. Provence(Marseille), 1977.

Felix Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Math. Ann.*, 65(4):435–505, 1908. ISSN 0025-5831. doi:10.1007/BF01451165.

Gerhard Hessenberg. *Grundbegriffe der Mengenlehre*, volume 1. Vandenhoek & Ruprecht, Göttingen, 1906. URL http://reader.digitale-sammlungen.de/de/fs1/object/display/bsb11171763_00002.html.

Graham Higman. Ordering by divisibility in abstract algebras. *Proc. Lond. Math. Soc. (3)*, 2:326–336, 1952. ISSN 0024-6115. doi:10.1112/plms/s3-2.1.326.

Ernst Erich Jacobsthal. Über den Aufbau der transfiniten Arithmetik. *Math. Ann.*, 66:145–194, 1909. ISSN 0025-5831. doi:10.1007/BF01450583. URL https://eudml.org/doc/158375.

Pierre Jullien. *Contribution à l'étude des types d'ordres dispersés.* PhD thesis, Université de Marseille, June 1968. 116p.

Georges Kurepa. *Ensembles ordonnées et ramifiés.* NUMDAM, [place of publication not identified], 1935. URL http://www.numdam.org/item?id=THESE_1935__175__1_0.

Richard Joseph Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971. ISSN 0003-486X. doi:10.2307/1970754.

Richard Joseph Laver. An order type decomposition theorem. *Ann. of Math. (2)*, 98:96–119, 1973. ISSN 0003-486X. doi:10.2307/1970907. URL https://doi.org/10.2307/1970907.

Carlos Martinez-Ranero. Well-quasi-ordering Aronszajn lines. *Fund. Math.*, 213(3):197–211, 2011. ISSN 0016-2736,1730-6329. doi:10.4064/fm213-3-1. URL https://doi.org/10.4064/fm213-3-1.

Justin Tatch Moore. A five element basis for the uncountable linear orders. *Ann. of Math. (2)*, 163(2):669–688, 2006. ISSN 0003-486X. doi:10.4007/annals.2006.163.669. URL https://doi.org/10.4007/annals.2006.163.669.

Justin Tatch Moore. A universal Aronszajn line. *Math. Res. Lett.*, 16(1):121–131, 2009. ISSN 1073-2780. doi:10.4310/MRL.2009.v16.n1.a12. URL https://doi.org/10.4310/MRL.2009.v16.n1.a12.

Saharon Shelah. Decomposing uncountable squares to countably many chains. *J. Combinatorial Theory Ser. A*, 21(1):110–114, 1976. ISSN 0097-3165. doi:10.1016/0097-3165(76)90053-4. URL https://doi.org/10.1016/0097-3165(76)90053-4.

Saharon Shelah. Can you take Solovay's inaccessible away? *Israel J. Math.*, 48(1):1–47, 1984. ISSN 0021-2172. doi:10.1007/BF02760522. URL http://dx.doi.org/10.1007/BF02760522.

Samir Siksek. Every integer greater than 454 is the sum of at most seven positive cubes. *Algebra Number Theory*, 10(10):2093–2119, 2016. ISSN 1937-0652,1944-7833. doi:10.2140/ant.2016.10.2093. URL https://doi.org/10.2140/ant.2016.10.2093.

Stevo Todorčević. Trees and linearly ordered sets. In *Handbook of set-theoretic topology*, pages 235–293. North-Holland, Amsterdam, 1984.