The universe constructed from a set (or class) of regular cardinals

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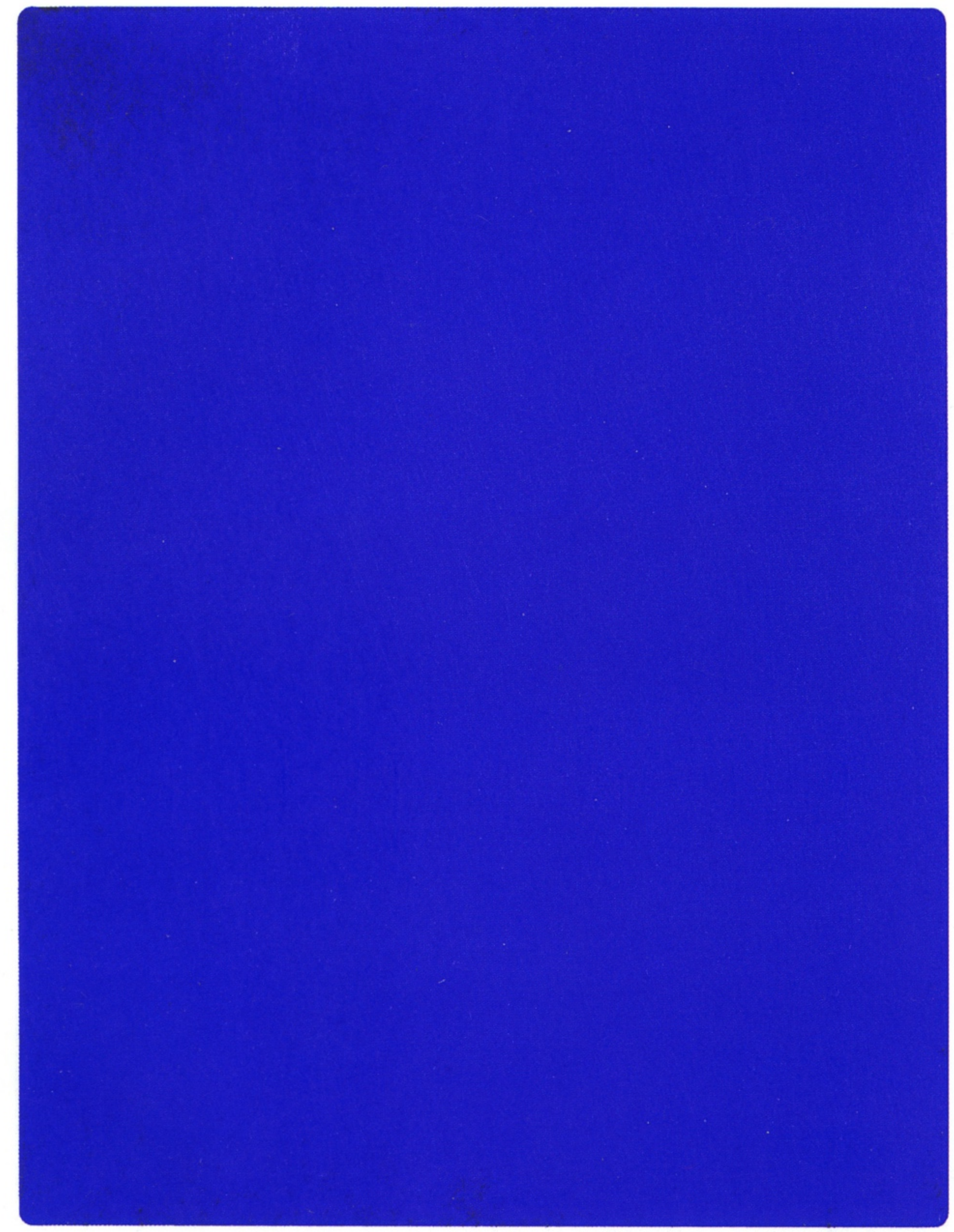
Philip Welch, University of Bristol



Part I: Background: $L[P]$ for a C.U.B. class $P \subseteq On$.
The Härtig Quantifier Model $C(I)$.

Part II: From $L[Card]$ to $L[Reg]$, and $L[S]$ for $S \subseteq Reg$.
The Regularity Quantifier Model $C(R)$.

• Consider c.u.b classes of ordinals $P \subseteq On$ and the universes $L[P]$ constructed from them.

Examples: $L[C^n]$ where $C^n =_{df} \{\alpha \mid (V_\alpha, \in) \prec_{\Sigma_n} (V, \in)\}$.

$L[I]$ where I is the class of *uniform Silver indiscernibles* thus:

$$I = \bigcap_{r \subseteq \omega;\, r^\sharp \text{exists}} I^r.$$

$L[Card]$ where $Card$ is the class of uncountable cardinals.

$$\vdots$$

- What do these models have in common, if anything?

- What are their properties? Are they models of GCH?
  What is the descriptive set-theoretic complexity of their reals?

Assuming only modest large cardinals in V (below a measurable with Mitchell order > 0):

- These models all have the same reals:

$$\mathbb{R}^{L[C^{23}]} = \mathbb{R}^{L[I]} = \mathbb{R}^{L[Card]} = \cdots$$

- In fact they are all elementary equivalent:

$$\langle L[C^{17}], \in, C^{17} \rangle \equiv \langle L[I], \in, I \rangle \equiv \langle L[Card], \in, Card \rangle \cdots$$

where the elementary equivalence is in the language $\mathcal{L}_{\dot{\in},\dot{P}}$ with a predicate symbol $\dot{P}$ for ordinals.

The reason behind this

- O^k is the sharp for the least inner model with a proper class of measurable cardinals. "O^k" is "O^{kukri}"

Theorem 1 (ZFC) *Suppose O^k exists. There is a definable proper class $C \subseteq On$ that is cub beneath every uncountable cardinal, so that for any definable cub subclasses $P, Q \subseteq C$:*

$$\mathbb{R}^{L[P]} = \mathbb{R}^{L[Q]}; \quad \langle L[P], \in, P\rangle \equiv \langle L[Q], \in, Q\rangle$$

where the elementary equivalence is in the language $\mathcal{L}_{\dot{\in},\dot{P}}$ with a predicate symbol $\dot{P}$. Moreover this theory is invariant into outer models of V, i.e. *into ZFC-preserving extensions.*

Slogan:

We are seeing if large cardinals affect the informational content of $L[Card]$.

The conclusion is that they do not: once we get to O^k these models become in one sense the same.

Definition 1 *Let O^k name the least sound active mouse of the form $M_0 =_{\mathrm{df}} \langle J^{E^{M_0}}_{\alpha_0}, E^{M_0}, F_0 \rangle$ so that*

$$M_0 \models \text{“}F_0 \textit{ is a normal measure on } \kappa_0 \wedge \exists \textit{ arbitrarily large measurable cardinals below } \kappa_0.\text{”}$$

(i) M_0 is a countable structure.

(ii) We may form iterated ultrapowers of M_0 repeatedly using the top measure F_0 and its images to form iterates $M_\iota =_{\text{df}} \langle J^{E_{M_\iota}}_{\alpha_\iota}, E_{M_\iota}, F_\iota \rangle$ so that $M_\iota \models$ "F_ι is a normal measure on κ_ι".

(iii) These iterations generate, or "leave behind", an inner model

$$L[E_0] =_{\text{df}} \bigcup_{\iota \in On} H^{M_\iota}_{\kappa_\iota} = \bigcup_{\iota \in On} H^{M_\iota}_{\kappa^+_\iota}.$$

(iv) The cub class of critical points $C_{M_0} = \langle \kappa_\iota \mid \iota \in On \rangle$ forms a class of indiscernibles that is cub beneath each uncountable cardinal, for the inner model $L[E_0]$.

(v) $L[E_0]$ is similarly the *minimal inner model of a proper class of measurables*: any other such is a simple iterated ultrapower model of $L[E_0]$.

- We iterate $L[E_0]$, or equivalently $O^k = M_0$, so that in the resulting model $L[E^C]$ ($C = Card$) the measurables are precisely the μ_α below.

Define the function:

$$c(\alpha) = \langle \aleph_{\omega\alpha+k} \mid 0 < k < \omega \rangle$$

and let

$$\mu_\alpha =_{\mathrm{df}} \aleph_{\omega\alpha+\omega}.$$

- Moreover in $L[E^C]$ the full measure on μ_α is generated by $c(\alpha)$.

More general P

Definition 1 *We say P is* appropriate *if it is any c.u.b. subclass of*

$$C_{M_0} =_{\mathrm{df}} \{\kappa_\alpha \mid \alpha \in On\}.$$

Let $\langle \lambda_\iota | \iota \in On \rangle$ be P's increasing enumeration. Define the function:

$$c(\alpha) = c^P(\alpha) = \langle \lambda_{\omega\alpha+k} \mid 0 < k < \omega \rangle$$

and

$$\mu_\alpha = \mu^P_\alpha =_{\mathrm{df}} \lambda_{\omega\alpha+\omega}.$$

Definition

For $\nu = \kappa_\nu \in C_{M_0}$ let $\mathbb{P}^\nu = \mathbb{P}^{P,\nu}$ be the following set of function pairs $\langle h, H \rangle$:

(i) $\boxed{H \in \Pi_{\alpha<\nu} U_\alpha,\ \mathrm{dom}(h) = \nu}$ and $\boxed{\mathrm{supp}(h) \text{ is finite}}$ where:
$\mathrm{supp}(h) =_{df} \{\alpha \in \mathrm{dom}(h) \mid h(\alpha) \neq \emptyset\}$.
(ii) [Various usual Prikry like conditions]

For $\langle f, F \rangle, \langle h, H \rangle \in \mathbb{P}^\nu$ set

$$\langle f, F \rangle \leq \langle h, H \rangle \text{ iff } \forall \alpha < \nu (f(\alpha) \supseteq h(\alpha) \wedge f(\alpha) \backslash h(\alpha) \subseteq H(\alpha)).$$

We let G^ν be $\mathbb{P}^\nu$-generic over $L[E^P]$, and we define $c = c_{G^\nu}$ by

$$c(\alpha) = \bigcup \{h(\alpha) \mid \exists H \langle h, H \rangle \in G^\nu\} \text{ for all } \alpha < \nu.$$

- $\mathbb{P}^\nu$ has the ν^+- c.c. (and this is best possible).

Theorem (Mathias Condition - Fuchs)

A function d is $\mathbb{P}^\nu$-generic over $L[E^P] \Leftrightarrow$

$$\forall X \in \prod_{\alpha<\nu} U_\alpha \cap L[E^P] \quad \bigcup_{\alpha<\nu} (d(\alpha)\backslash X(\alpha)) \textit{ is finite.}$$

(Here U_α is on μ_α, the α'th measurable of $L[E^P]$.)

Definition

A sequence $\vec{c} = \langle c(\alpha) \mid \alpha \in \Delta\rangle$ where Δ is a set of measurable cardinals, with U_α a normal measure on α, is said to have the *$\vec{U}$-set property* if for every sequence $\vec{A} = \langle A_\alpha \mid \alpha \in \Delta\rangle$ with each $A_\alpha \in U_\alpha$, then

$$\bigcup_{\alpha\in\Delta} (c(\alpha)\backslash A_\alpha) \text{ is finite.}$$

- If $p = \langle h, H\rangle \in L[E^P]$, define $d(\alpha) = h(\alpha) \cup (c(\alpha) \cap H(\alpha))$. Thus we have a $d \in L[E^P][c]$ and $L[E^P][c] = L[E^P][d]$.

G. Fuchs, "*A Characterisation of Generalized Příkrý forcing*", Archive for Math. Logic, 2005.

Corollary

Let c be $\mathbb{P}^\nu$-generic over $L[E^P]$. Let $p \in \mathbb{P}^\nu$. Then there exists a sequence d which is $\mathbb{P}^\nu$-generic over $L[E^P]$ so that:
(i) $|\bigcup_{\alpha<\nu}(c(\alpha)\triangle d(\alpha))| < \omega$;
(ii) $p \in G_d$.

Consequently we have also:

Corollary (Weak Homogenity)

If $\varphi(v_0, \ldots, v_{n-1})$ is any formula and $\check{a}_1, \ldots \check{a}_{n-1}$ any forcing names for elements of $L[E^P]$, and $p \in \mathbb{P}^\nu$ we have

$$p \Vdash_{\mathbb{P}^\nu} \varphi(\check{a}_1, \ldots, \check{a}_{n-1}) \Rightarrow \mathbb{1} \Vdash_{\mathbb{P}^\nu} \varphi(\check{a}_1, \ldots, \check{a}_{n-1}).$$

- If $p = \langle h, H \rangle \in L[E^P]$, define $d(\alpha) = h(\alpha) \cup (c(\alpha) \cap H(\alpha))$. Thus we have a $d \in L[E^P][c]$ and $L[E^P][c] = L[E^P][d]$.

The class version: the full forcing $\mathbb{P}^\infty = \mathbb{P}^P$

If $\nu \in D =_{\mathrm{df}} \{\nu \in C \mid \nu = \lambda_\nu\}$, the top measurable of M_ν, we have $\mathbb{P}^\nu \in \Delta_1^{M_\nu}$. Then:

$$c^\nu \text{ is } \mathbb{P}^\nu\text{-generic over } L[E^C] \Longleftrightarrow c^\nu \text{ is } \mathbb{P}^\nu\text{-generic over } H_{\nu^+}^{L[E^C]}$$

(1) "Stretch" $H^\nu =_{\mathrm{df}} H_{\nu^+}^{L[E^C]}$ to $H_\infty =_{\mathrm{df}} H_{On^+}^{\text{``}L[E^C]\text{''}}$.

(2) For $\iota, \nu \in D$, $\iota < \nu$, $\widetilde{\pi}_{\iota,\nu} : \langle H^\iota, \mathbb{P}^\iota, \Vdash_\iota \rangle \longrightarrow_e \langle H^\nu, \mathbb{P}^\nu, \Vdash_\nu \rangle$.

(3) $\langle H^\infty, E, \Vdash_\infty, \mathbb{P}^\infty, \langle \widetilde{\pi}_{\iota,\infty} \rangle \rangle =_{\mathrm{df}} \mathrm{Lim}_{\iota \to \infty, \iota \in D} \langle H^\iota, \in, \Vdash_\iota, \mathbb{P}^\iota, \langle \widetilde{\pi}_{\iota,\nu} \rangle \rangle$.

• Note: $\mathbb{P}^\infty$ does not have the On-c.c. H^∞ will be a natural Kelley-Morse model: but $\mathbb{P}^\infty$ is still a class forcing over this model.

- The definability of the forcing $\mathbb{P}^{\nu}$ over $H_{\nu^+}^{L[E^P]}$ for $\nu \in D$ together with

(i) $L_{\nu}[E^P] \prec L[E^P]$; *and*

(ii) its weak homogeneity,

yields the definability of the theory of $L[E^P][c]$ over any such $H_{\nu^+}^{L[E^P]}$.

Theorem

Assume that O^k exists and P is an appropriate class.
(i) $K^{L[P]} = L[E^P]$ where E^P is a coherent filter sequence so that

$$L[E^P] \models \text{“ } \kappa \textit{ is measurable”} \Leftrightarrow \kappa = \mu_\alpha \textit{ for some } \alpha.$$

(ii) The class $c^P =_{df} \langle c^P(\alpha) \mid \alpha \in On \rangle$ of ω-sequences is mutually $\mathbb{P}^P$-generic over $L[E^P]$ for the full product Prikry forcing $\mathbb{P}^P$; moreover

$$\boxed{L[P] = L[E^P][c^P] = L[c^P].}$$

Secondary Statement of Main Theorem

Corollary 1 *Assume O^k exists. Let P be any appropriate class. Then in $L[P]$:*
(i) Each μ_α is Jónsson, and c_α forms a coherent sequence of Ramsey cardinals *below μ_α. But there are no measurable cardinals.*
(ii) For any $L[P]$-cardinal κ we have $\diamondsuit_\kappa$, $\square_\kappa$, $(\kappa, 1)$-morasses etc. etc.
(iii) The GCH holds but $V \neq HOD$.
(iv) There is a Δ^1_3 wellorder of $\mathbb{R} = \mathbb{R}^{K^{L[P]}}$;
$Det(\alpha\text{-}\mathbf{\Pi}^1_1)$ holds for any countable α, but $Det(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1))$ fails (Simms, Steel).

Part II: Going to $L[Reg]$

- $O^s = O^{sword}$ is the least inner mouse whose top measure concentrates on the measures below.

We form an iteration of $M_0 = O^s$ in blocks:

(1) iterate the least measurable of M_0 to align onto $\aleph_\omega$ now in the model $M_{\aleph_\omega}$; then the least measurable of $M_{\aleph_\omega}$ above $\aleph_\omega$ to align onto $\aleph_{\omega\cdot 2}$ now in the model $M_{\aleph_{\omega\cdot 2}}$;

(2) If V has, *e.g.*, unboundedly many 1-inaccessibles, then there will be inaccessible stages λ where in M_λ λ is the image of critical points from below, arising from our alignment process. In this case we use the order zero measure on λ to form the ultrapower $M_\lambda \longrightarrow M_{\lambda+1}$.
We then iterate the least measure which has now appeared in $M_{\lambda+1}$ above λ up to the next simple $\aleph_{\tau+\omega}$.

Leaving measures behind

(3) If λ is of the form $\rho^\lambda_\omega =_{df} sup\langle \rho^\lambda_k | k < \omega\rangle$ where $\pi_{\rho^\lambda_k, \rho^\lambda_{k+1}}(\rho^\lambda_k) = \rho^\lambda_{k+1}$ with $\rho^\lambda_k \in Inacc$, then use the next measure above λ in M_λ (if such exists); or else the order 1 measure of M_λ, to iterate up to the next simple limit $\aleph$.

However, here we have:

$\pi_{\rho^\lambda_k, \rho^\lambda_{k+1}}(E_{\rho^\lambda_k}) = E_{\rho^\lambda_{k+1}}$
And thus: $\pi_{\rho^\lambda_k, \rho^\lambda_\omega}(E_{\rho^\lambda_k})$ on $\lambda = \rho^\lambda_\omega$, is the measure that is left behind on λ.

(4) Otherwise: then $\lambda \in SingCard$, and not a simple limit $\aleph$, so then we finish as in (2) iterating the next unused measure to the next simple limit $\aleph_{\tau+\omega}$.

The upshot is that we have a model $L[E^R]$ ($R = Reg$) with: μ measurable in $L[E^R]$ iff

Either:
$\mu = \mu_\alpha = \aleph_{\omega\cdot\alpha+\omega}$ for some α and the measure is generated by $\langle \aleph_{\omega\cdot\alpha+k} \rangle_{k<\omega}$.

Or:
$\mu = \mu_\alpha = \rho^\alpha_\omega$ for some $\alpha = \sup\{\rho^\alpha_k\}_{k<\omega}$ and the measure is generated by inaccessibles $\langle \rho^\alpha_k \rangle_{k<\omega}$.

But also:

Lemma

All but at most finitely many V-inaccessibles are of the form ρ_n^α for some n, α.

Magidor genericity

To deduce Magidor genericity of the $\vec{c}$ sequence needs a recent result of Ben-Neria.

Definition

Let $\vec{c}$ be a set of ω-sequences with $c(\alpha) \subseteq \alpha$. Then $\vec{c}$ has the *(strict) separation property* if only finitely many (respectively no) pairs of the form $\langle \nu, \kappa \rangle$ and $\langle \nu', \kappa' \rangle$ with $\nu \in c(\kappa), \nu' \in c(\kappa')$ are *interleaved*, that is satisfy $\nu \leq \nu' < \kappa < \kappa'$.

Theorem (Ben Neria)

If $\forall\nu \in \mathit{Inacc}: \; G \restriction \nu =_{df} \langle c(\alpha) \mid \alpha < \nu\rangle$ *has both the* $\vec{U}_\alpha$*- Set and then Separation properties then:*

$$G \restriction \nu \textit{ is } \mathbb{P}_\nu\textit{-Magidor-generic over } L[\vec{U}^R].$$

- Here $L[\vec{U}^R]$ is the least Kunen-style inner model constructed from the measure sequence $U_\alpha =_{df} E^R_{\mu_\alpha}$ where the latter $E^R_{\mu_\alpha}$ are the full measures of $L[E^R]$.

- The model $L[\vec{U}^R]$ actually is also an $L[E]$-model, call it $L[E_0^R]$ which has the same measurables as $L[E^R]$. It is just that our original iteration may not pick out the *least* inner model with exactly those measurables.

(Compare: there are fine-structural $L[E]$-models with precisely one measurable cardinal, but that does not mean that $L[E]$ is the least such - which is of the form $L[\mu]$.)

Corollary

$O^{\text{sword}} \notin L[\text{Reg}]$.

We have conversely:

Lemma

Suppose O^{sword} *exists. Then it is consistent that it is the* $<^*$*-least mouse not in* $L[\text{Reg}]$*. Consequently it is consistent that the structure of* Reg *is such that the construction procedure above cannot be effected by any smaller mouse* $N_0 <^* O^{\text{sword}}$.

This will be a special case of the next result.

Theorem

(a) ZFC $\vdash$*" Let* $S_1 \subseteq$ Reg *be a set or proper class of infinite regular cardinals. Then* $O^{\text{sword}} \notin L[S_1]$*".*
(b) Both these results are best possible. In particular for (a) O^s *cannot be replaced by any sound mouse* $M <^* O^s$.

Corollary (to the argument)

If On *is Mahlo, then* O^s*, if it exists, is* $<^*$*-least not in* $L[\mathrm{Reg}]$ *and consequently we must use* O^s *and nothing smaller to generate an inner model* W *with* $L[\mathrm{Reg}] = W[\vec{c}]$.

The Härtig quantifier I

Definition

$\mathcal{M} \models \mathsf{I}xy\, \varphi(x,\vec{p})\psi(y,\vec{p}) \leftrightarrow$

$$|\{a \mid \mathcal{M} \models \varphi[a,\vec{p}]\}| = |\{b \mid \mathcal{M} \models \psi[b,\vec{p}]\}|$$

$$\begin{array}{lcr} L_0^{\mathsf{I}} & = & \emptyset \\ L_{\alpha+1}^{\mathsf{I}} & = & Def_{\mathcal{L}^{\mathsf{I}}}(L_\alpha^{\mathsf{I}}) \\ L_\lambda^{\mathsf{I}} & = & \bigcup_{\alpha<\lambda} L_\alpha^{\mathsf{I}} \end{array}$$

and then $L^{\mathsf{I}} = \bigcup_{\alpha \in On} L_\alpha^{\mathsf{I}}$.

- Then L^{I} is the *Härtig quantifier model* of [KMV], there written $C(I)$.

[KMV] J. Kennedy, M. Magidor, J. Väänänen *"Inner Models from Extended Logics"* to appear.

When $P = Card$

Lemma 1 $C(I)\,(= L^{\mathrm{I}}) = L[Card]$.

Theorem

$\neg O^k \iff K^{C(I)} = K.$

Corollary

$(V = L[E]) \quad \neg O^k \iff V = C(I).$

The Regularity quantifier R

Definition

$\mathcal{M} \models \mathsf{R}x\,\varphi(x,\vec{p}) \quad \Leftrightarrow \quad |\{a \mid \mathcal{M} \models \varphi[a,\vec{p}]\}| \in Reg.$

$$\begin{array}{rcr} L_0^{\mathsf{R}} & = & \emptyset \\ L_{\alpha+1}^{\mathsf{R}} & = & Def_{\mathcal{L}^{\mathsf{I}}}(L_\alpha^{\mathsf{R}}) \\ L_\lambda^{\mathsf{R}} & = & \bigcup_{\alpha<\lambda} L_\alpha^{\mathsf{R}} \end{array}$$

and then $L^{\mathsf{R}} = \bigcup_{\alpha \in On} L_\alpha^{\mathsf{R}}$.

When $P = Card$

Lemma

$C(R)\ (= L^{\mathsf{R}}) = L[Reg]$.

Theorem

$\neg O^s \iff K^{C(R)} = K$.

Corollary

$(V = L[E]) \quad \neg O^s \iff V = C(R)$.