Maximal Prikry-Type Sequences

Ben-Zion Weltsch

Rutgers University

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Comparing Generic Extensions

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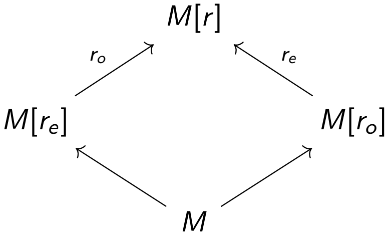
Example

Let r be a Cohen real, r_e the even digits and r_o the odd digits. Then r_e and r_o are also generic for the Cohen forcing.

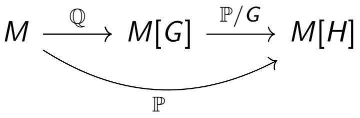


And in fact $M[r] = M[r_e \times r_o]$ and $M[r_e] \cap M[r_o] = M$.

More examples

Intermediate Models

Let $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$. Then $\mathbb{P}$ *projects* onto $\mathbb{Q} = \mathrm{Col}(\omega_1, 2^\omega)$.



Projections

$\pi : \mathbb{P} \to \mathbb{Q}$ is a *projection* if π is order-preserving and for all $p \in \mathbb{P}$ and $q \leq \pi(p)$ there is a $p' \leq p$ such that $\pi(p') \leq q$.

If $H \subseteq \mathbb{P}$ is generic then $\pi``H$ generates a generic filter for $\mathbb{Q}$.

Prikry Forcing

Let κ be a measurable cardinal and U a normal ultrafilter on κ.

Prikry Forcing

Prikry forcing $\mathbb{P}_U$ consists of pairs (s, A) where $s \in [\kappa]^{<\omega}$ (the *stem*) and $A \in U$ (the *constraint*). $(s, A) \leq (t, B)$ iff

1. $t \sqsubseteq s$ (i.e. s end-extends t)
2. $s \setminus t \subseteq B$
3. $A \subseteq B$

Let $G \subseteq \mathbb{P}$ be generic and $C_G = \bigcup\{s : \exists A\,(s, A) \in U\}$. Then $\sup C_G = \kappa$ and $|C_G| = \omega$ (so $\mathrm{cf}(\kappa) = \omega$). We call C_G a *Prikry sequence*, and in fact $V[G] = V[C_G]$.

Prikry Facts

Direct Extensions

If $p \leq q$ with the same stem, then p is a *direct extension* of q, written $p \leq^* q$.

Prikry Property

Given any φ in the forcing language and $p \in \mathbb{P}$, there is $p^* \leq^* p$ such that $p^* \parallel \varphi$.

Mathias Criterion

Let $W \supseteq V$ be a forcing extension by $\mathbb{P}_U$. Then $C \in W$ is a Prikry sequence iff for all $A \in U$, $C_n \in A$ for all but finitely many n.

Comparing Prikry Sequences

The Mathias criterion is used to prove:

Folklore

If C, C' are two Prikry sequences over V then $|C \Delta C'| < \omega$ iff $V[C] = V[C']$.

Gitik-Kanovei-Koepke use this fact to prove a structure theorem for intermediate models of Prikry forcing.

Gitik-Kanovei-Koepke [3]

If C is a Prikry sequence over V then for all $X \in V[C]$, there is a $C' \subseteq C$ so that $V[C'] = V[X]$.

Corollary

Every intermediate model of a Prikry forcing extension is again a Prikry forcing extension, with the same ultrafilter.

Maximality of Prikry Sequences

So the following property seems to have strong consequences for intermediate models of Prikry extensions.

The Maximality Property

Let $\mathbb{P}$ be a Prikry-type forcing. We say $\mathbb{P}$ has the *maximality property* if for any two $\mathbb{P}$-Prikry sequences C and C', $V[C] = V[C']$ if and only if $|C \Delta C'| < \omega$.

This is a maximality property since the Prikry sequence C is maximal among all other Prikry sequences in $V[C]$, with respect to $\subseteq$ mod finite.
In particular, we will investigate *supercompact* Prikry forcing.

Supercompactness

Let $\kappa < \lambda$ be regular.

1. $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$
2. For $x, y \in P_\kappa(\lambda)$, $x \prec y$ iff $x \subseteq y$ and $|x| < |\kappa \cap y|$.

Let U be an ultrafilter on $P_\kappa(\lambda)$.

1. U is *fine* if for all $\alpha < \lambda$, $\{x \in P_\kappa(\lambda) : \alpha \in x\} \in U$
2. U is *normal* if whenever $f : P_\kappa(\lambda) \to P_\kappa(\lambda)$ is such that $f(x) \prec x$ for almost all x in U, f is constant on a set in U. Equivalently, U is closed under *diagonal intersections*: Given $\{A_x \subseteq P_\kappa(\lambda) : x \in P_\kappa(\lambda)\}$ we have

$$\Delta_x A_x = \{y \in P_\kappa(\lambda) : x \prec y \Rightarrow y \in A_x\} \in U$$

κ is *supercompact* if for all $\lambda > \kappa$ there is a κ-complete, normal, fine ultrafilter (a *supercompactness measure*) on $P_\kappa(\lambda)$.

Supercompact Prikry Forcing

Supercompact Prikry was first used by Magidor to prove the consistency of $\neg\mathrm{SCH}_{\aleph_\omega}$ and has seen many other applications. Let U be a normal, fine ultrafilter on $P_\kappa(\lambda)$.

Supercompact Prikry

A condition in $\mathbb{P}_U$ is a pair (s, A) where $s \in [P_\kappa(\lambda)]^{<\omega}$ (the *stem*) that is increasing with respect to $\prec$, and $A \in U$. $(s, A) \leq (t, B)$ iff

1. $t \sqsubseteq s$ (i.e. s end-extends t)
2. $s \setminus t \subseteq B$
3. $A \subseteq B$

Now the Prikry sequence is $\langle x_n \in P_\kappa(\lambda) : n \in \omega\rangle$, and $\langle \sup(x_n) : n < \omega\rangle$ singularizes λ. In fact every α between κ and $|\lambda^{<\kappa}|$ is singularized.

Note that the order $\prec$ is *not* total, which causes much difficulty.

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Conjecture (Woodin, 90's)

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The key is this lemma, generalizing a result of Benhamou [4].

Separation Lemma

Let U be a normal, fine ultrafilter on $P_\kappa(\lambda)$ and let C be a $\mathbb{P}_U$-Prikry sequence. Suppose $C' \in V[C]$ such that $C' \subseteq P_\kappa(\lambda)$ countable yet $C' \cap C = \emptyset$. Then there is an $A \in U$ disjoint from C'.

If maximality fails, in $V[C]$ there would be another Prikry sequence C' almost disjoint from C. By Mathias, we may assume C and C' are actually disjoint. But then C' is disjoint from a set in U, violating Mathias' criterion.

Lemma Proof Sketch

A key to the in the proof of the lemma is the strong Prikry property:

Strong Prikry Property

Let $p \in \mathbb{P}_U$ and let D be open dense. Then there is some $n \in \omega$ and $p^* \leq^* p$ so that every n-step extension of p^* is in D.

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By the completeness of U and closure of $\leq^*$, it suffices to show the following:

Claim

Whenever $(s, A) \Vdash \dot{x} \notin \dot{C}$, there is some $(s, B) \leq^* (s, A)$ such that $(s, B) \Vdash \dot{x} \notin B$.

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- But $q \Vdash \alpha_t = \alpha^* \notin \dot{x}$
- Hence $q \Vdash \dot{x} \notin B$, a contradiction. $\square$

Intermediate Models of Supercompact Prikry

Maximality of supercompact Prikry provides some structure of intermediate models. But in contrast with regular Prikry, there is a variety of intermediate extensions.

Theorem (Gitik [5])

Suppose $\mathbb{Q}$ is κ-distributive. Let $\lambda = 2^{|\mathbb{Q}|}$. If U is a normal, fine measure on $P_\kappa(\lambda)$ then $\mathbb{P}_U$ projects onto $\mathbb{Q}$.

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4. Adding a κ-Kurepa tree
5. Many others

Non-Maximality: Product Forcing

Products of Prikry forcing also fail to have the maximality property, and in fact add Cohen reals.

Fact

Let U_0, U_1 be uniform ultrafilters on κ. Then there is an ultrafilter W so that $\mathbb{P}_{U_0} \times \mathbb{P}_{U_1} \cong \mathbb{P}_W \cong \mathbb{P}_W \times \mathrm{Add}(\omega, 1)$.

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5. Let $G \subseteq \mathbb{P}_{U_0} \times \mathbb{P}_{U_1}$. Working in $V[G]$: $f(n) =$ the unique i so that $(\dot{\alpha}_n)_G$ is in the U_i Prikry sequence.

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6. $\langle (s, A), (t, B) \rangle \mapsto \langle (s \cup t, A \cup B), f \restriction |s \cup t| \rangle$ is an isomorphism.

Non-Maximality: A conjecture

In fact, all examples we know of non-maximality for the standard Prikry forcing add bounded subsets to κ.
On this slide, let U be a κ-complete ultrafilter on κ.

Theorem (Devlin)

$\mathbb{P}_U$ adds bounded subsets of κ iff U is not Rowbottom.

This connects to the Rudin-Kiesler order.

Folklore Fact

U is Rowbottom iff U is $<_{RK}$ minimal among uniform κ-complete ultrafilters on κ.

In general, the examples of U with the maximality property are $<_{RK}$-minimal, and the ones without the maximality property are not $<_{RK}$-minimal.

Non-Maximality: Product Measures

Here, $\mathbb{P}_U^T$ denotes *tree* Prikry forcing, a generalization of Prikry. $\mathbb{P}_U^T$ is much better behaved than the classical Prikry forcing, as it never adds bounded subsetes of κ if U is κ-complete. Furthermore, $\mathbb{P}_U^T \approx \mathbb{P}_U$ whenever U is Rowbottom.

Fact (Hamkins)

For any ultrafilters U and V, $\mathbb{P}_{U\times V}^T$ does not have the maximality property.

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For any ultrafilters U and V, $\mathbb{P}_{U\times V}^T$ does not have the maximality property.

This follows from the analysis of Prikry forcing and iterated ultrapowers. This analysis of iterated ultrapowers gives more evidence for our conjecture that U has the maximality property iff U is $<_{RK}$-minimal.

Prikry and Iterated Ultrapowers

Bukovsky and Dehornoy connected Prikry forcing to iterated ultrapowers.

Theorem (Bukovsky and Dehornoy)

Let $j_U^\omega : V \to M_\omega$ be ωth iterated ultrapower of V by U. Then the critical sequence $\langle \kappa, j(\kappa), j(j(\kappa)), \ldots \rangle$ is $\mathbb{P}_{j_\omega(U)}$-generic over M_ω.

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Hamkins' analysis of the maximality property is via an extension of this analysis.

Theorem (Hamkins)

$\mathbb{P}^{T}_{U}$ has the maximality property if and only if U admits no *non-canonical seed* via j^{U}_{ω}.

Hence the maximality property has interesting consequences for iteration theory of supercompactness measures.

Canonical Seeds

Here we define seeds:

Definitions

1. a is a *seed* for U via j if $X \in U \iff a \in j(X)$.
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2. The *canonical seed sequence* for U is the sequence $\langle j^U_{n+1,\omega}([\text{id}]_{U_n}) : n < \omega\rangle$

Hamkins's theorem says that $\mathbb{P}^T_U$ has the maximality property iff the only seeds for U via j^U_ω are the ones on the canonical seed sequence, i.e. there are no non-canonical seeds.
The only ultrafilters we know of that admit non-canonical seeds are not $<_{RK}$-minimal, providing more evidence for our conjecture.
When U does not concentrate on ordinals, the canonical seed sequence becomes quite difficult to analyze.

Canonical Seeds and Generators

Example: Product Measures

- Let U be a normal ultrafilter on κ and $\langle \kappa_n : n < \omega \rangle$ be the critical sequence.
- The canonical seed sequence for U^2 is $\langle (\kappa_{2n}, \kappa_{2n+1}) : n \in \omega$.
- But any pair (κ_i, κ_j) is a seed for U^2 via $j_\omega^{U^2}$, showing maximality fails.

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Generators

An ordinal ξ is a *generator* of an embedding $j : V \to M$ if $\xi \notin H^M(j[V] \cup \xi)$.

ξ is a generator of j_U if, roughly, whenever we derive ultrafilters W_0, W_1 from j_U using $\zeta < \xi$ as seeds respectively, $W_0 <_{RK} W_1$.

Generators and Maximality

There also seems to be a connection between the number of generators of j_U and the maximality property for $\mathbb{P}_U^T$. The following holds for all known examples of U with the maximality property, and fails for all known U without it:

Minimal Generators Property

Let U be a σ-complete ultrafilter on X. Say $|X| = \lambda$. We say U has the *minimal generators property* if $\sup j_U[\lambda]$ is the largest generator of j_U.

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Minimal Generators Property

Let U be a σ-complete ultrafilter on X. Say $|X| = \lambda$. We say U has the *minimal generators property* if $\sup j_U[\lambda]$ is the largest generator of j_U.

This property partially characterizes supercompactness measures, and this property fails for strongly compact measures. So it seems very likely that maximality fails for strongly compact measures.

Further Results and Future Directions

Theorem (W. 2026)

Supercompact Magidor forcing has the maximality property.
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Let $\mathcal{U}_\kappa = \{U : U \text{ is } \sigma\text{-complete, uniform over } \kappa\}$. For any $U \in \mathcal{U}_\kappa$, $\mathbb{P}_U$ has the maximality property iff U is $<_{RK}$-minimal in $\mathcal{U}_\kappa$.

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Ongoing with Benhamou-Thei

Classify all intermediate models of super- and strongly-compact Prikry forcing.

Thanks!

Thank you for your attention!

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