Properties preserved in cofinal extensions

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> Psychologists assure us that tall people command more attention and respect than short ones. In as anthropomorphic a field as logic, it would follow that taller concepts excite more imagination than shorter ones. Thus, more is written about tall end extensions than stubby cofinal ones, and, asked for a preference between tall and short models, most logicians would make the tall choice. Such high-minded strategy might work well in the short run; but in the long run we must pay everything its due.
>
> C. Smoryński 1981

The preservation of induction in cofinal extensions

Definition

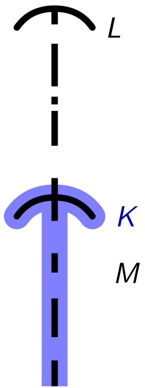
Let M, K be linearly ordered structures.

- We say K is a *cofinal extension* of M, and write $K \supseteq_{\mathrm{cf}} M$, if $K \supseteq M$ and

$$\forall y \in K \setminus M \quad \exists x \in M \quad x \geqslant y.$$

- We say L is an *end extension* of M, and write $L \supseteq_{\mathrm{e}} M$, if $L \supseteq M$ and

$$\forall y \in L \setminus M \quad \forall x \in M \quad x < y.$$



This talk

- elementarity
- induction
- failure of induction

Background

Fix $n \in \mathbb{N}$ throughout.

- $\mathcal{L}_\mathrm{A} = \{0, 1, +, \times, <\}$.
- A quantifier is *bounded* if it is of the form $\forall v < t$ or $\exists v < t$.
- An $\mathcal{L}_\mathrm{A}$ formula is Δ_0 if all its quantifiers are bounded.
- $\Sigma_n = \{\exists \bar{v}_1\ \forall \bar{v}_2\ \cdots\ \mathrm{Q}\bar{v}_n\ \theta : \theta \in \Delta_0\}$ and $\Pi_n = \{\forall \bar{v}_1\ \exists \bar{v}_2\ \cdots\ \mathrm{Q}'\bar{v}_n\ \theta : \theta \in \Delta_0\}$.
- A subset of an $\mathcal{L}_\mathrm{A}$ structure is Δ_{n+1}-*definable* if it is both Σ_{n+1}- and Π_{n+1}-definable.
- $\mathrm{I}\Gamma$ consists of the axioms of PA^- and for every $\theta \in \Gamma$,
$$\theta(0) \wedge \forall x\ (\theta(x) \to \theta(x+1)) \to \forall x\ \theta(x).$$
- $\mathrm{B}\Gamma$ consists of the axioms of $\mathrm{I}\Delta_0$ and for every $\theta \in \Gamma$,
$$\forall a\ (\forall x < a\ \exists y\ \theta(x, y) \to \exists b\ \forall x < a\ \exists y < b\ \ \theta(x, y)).$$
- exp asserts the totality of $x \mapsto 2^x$ over $\mathrm{I}\Delta_0$.
- (Paris–Kirby 1978) $\mathrm{I}\Delta_0 = \mathrm{I}\Sigma_0 \dashv \mathrm{B}\Sigma_1 \dashv \mathrm{I}\Sigma_1 \dashv \mathrm{B}\Sigma_2 \dashv \mathrm{I}\Sigma_2 \dashv \mathrm{B}\Sigma_3 \dashv \mathrm{I}\Sigma_3 \dashv \cdots$ and none of the converses holds. Also $\mathrm{I}\Delta_0 + \exp \nvdash \mathrm{B}\Sigma_1 + \exp \nvdash \mathrm{I}\Sigma_1$.
- (Parsons 1970; Parikh 1971) $\mathrm{I}\Sigma_1 \vdash \exp$ but $\mathrm{B}\Sigma_1 \nvdash \exp$.

Γ is a class of $\mathcal{L}_\mathrm{A}$ formulas

Elementarity

Fix $n \in \mathbb{N}$ throughout.

Definition

Let $M, K \models \mathrm{PA}^-$. Say K is an *n-elementary* extension of M, and write $K \succcurlyeq_n M$, if $K \supseteq M$ and for all $\theta \in \Sigma_n$ and $\bar{a} \in M$,

$$M \models \theta(\bar{a}) \quad \Leftrightarrow \quad K \models \theta(\bar{a}).$$

Theorem (Gaifman–Dimitracopoulos 1980; Kaye 1991)

Let $M, K \models \mathrm{PA}^-$ such that $M \subseteq_{\mathrm{cf}} K$. Then $M \preccurlyeq_{n+2} K$ if one of the following holds.

- $M \models \mathrm{B}\Sigma_{n+1}$ and $M \preccurlyeq_0 K$.
- $M \models \mathrm{B}\Sigma_{n+1} + \exp$ and $K \models \mathrm{I}\Delta_0 + \exp$.
- $M \models \mathrm{Coll}(\Sigma_{n+1})$ and $M \preccurlyeq_0 K$. If $n \geqslant 1$, then also $K \models \mathrm{Coll}(\Sigma_n)$.

$\mathrm{Coll}(\Sigma_n) = \mathrm{B}\Sigma_n - \mathrm{I}\Delta_0.$

Proposition (Lessan 1978)

There are models of $\mathrm{I}\Sigma_n$ that have 0-elementary cofinal proper extensions but have no $(n+2)$-elementary ones. (These are the pointwise Σ_{n+1}-definable models in which $\mathbb{N}$ is Σ_{n+1}-definable.)

Corollary (to the theorem above)

If $M \models \mathrm{B}\Sigma_{n+1}$ and $M \preccurlyeq_{0,\mathrm{cf}} K$, then $K \models \mathrm{I}\Sigma_n$. (Note $\mathrm{I}\Sigma_n \subseteq \Pi_{n+2}$.)

Preserving collection

Fix $n \in \mathbb{N}$ throughout.

Question (Chong 2017)

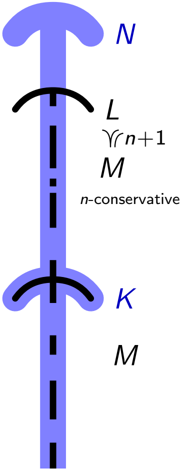
Let $M \models \mathrm{B}\Sigma_{n+1}$ and $K \succcurlyeq_{0,\mathrm{cf}} M$. Must $K \models \mathrm{B}\Sigma_{n+1}$?

Theorem (W)

Let $M \models \mathrm{I}\Sigma_{n+1}$ and $K \succcurlyeq_{0,\mathrm{cf}} M$. If $n \geqslant 1$, then $K \models \mathrm{B}\Sigma_{n+1}$.

Proof sketch

- Without loss of generality, assume M is countable.
- Build an $(n+1)$-elementary n-conservative (end) extension $L \supsetneq M$.
  Here *n-conservativity* means that for all Σ_n-definable $S \subseteq L$ and all $b \in M$,
  $$S \cap \{x \in M : x < b\} \quad \text{is coded in } M.$$
- (Kossak 1990) Amalgamate K and L over M into $N \succneqq_{n+1,\mathrm{e}} K$.
- (Paris–Kirby 1978) The existence of such an extension for K implies $K \models \mathrm{B}\Sigma_{n+1}$. □

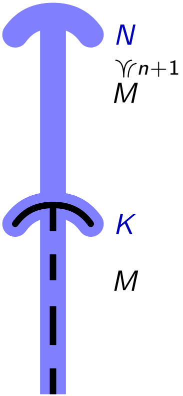


Achieving collection

Fix $n \in \mathbb{N}$ throughout.

Theorem (Paris 1981; Clote–Hájek–Paris 1990)

Every $M \models \mathrm{I}\Sigma_n$ has an $(n+1)$-elementary cofinal extension $K \models \mathrm{B}\Sigma_{n+1}$.



Proof sketch

- Use Compactness to obtain a non-cofinal $N \succcurlyeq_{n+1} M$.
- Split this extension into a cofinal extension K followed by an end extension. Then $M \preccurlyeq_{n+1} K \precneqq_n N \models \mathrm{I}\Delta_0$. Also $N \models \mathrm{I}\Sigma_{n-1}$ if $n \geqslant 1$.
- (Paris–Kirby 1978; Adamowicz–Clote–Wilkie 1985) The existence of such an extension for K implies $K \models \mathrm{B}\Sigma_{n+1}$. □

Remark

Paris's proof for countable models uses a coded ultrapower that is designed to make an arbitrarily chosen false instance of $\mathrm{B}\Sigma_{n+1}$ true.

Question

Is there an $(n+1)$-elementary proper cofinal extension of a model of $\mathrm{I}\Sigma_n + \neg\mathrm{B}\Sigma_{n+1}$ that is not elementary but does not turn any false instance of $\mathrm{B}\Sigma_{n+1}$ true?

Weak collection

Fix $n \in \mathbb{N}$ throughout.

Definition

$B\Sigma_{n+1}(1/2) + \exp$ consists of the axioms of $I\Sigma_n + \exp$ and for every $\theta \in \Sigma_{n+1}$,

$$\forall a \ (\forall x < a \ \exists y \ \theta(x, y) \to \exists b \ \exists^{1/2} x < a \ \exists y < b \ \theta(x, y)).$$

Fact (Belanger–Chong–Wang–W–Yang)

Replacing $1/2$ by any $r \in \mathbb{Q} \cap (0, 1)$ would give an equivalent theory.

Observation

$B\Sigma_{n+1} + \exp \vdash B\Sigma_{n+1}(1/2) + \exp \vdash I\Sigma_n + \exp.$

Theorem (Groszek–Slaman 1994)

$I\Sigma_n + \exp \nvdash B\Sigma_{n+1}(1/2) + \exp.$

If we apply the bottom theorem to $M \not\models B\Sigma_{n+1}(1/2) + \exp$, then $M \not\preccurlyeq_{n+2} K$.

Theorem (Belanger–Chong–Wang–W–Yang)

Every countable $M \models I\Sigma_n + \exp$ has an $(n + 1)$-elementary cofinal extension $K \models B\Sigma_{n+1}(1/2) + \exp$ that does not turn any false instance of $B\Sigma_{n+1}$ true.

Preserving cuts: failure of collection $\sim$ definable cut

Fix $n \in \mathbb{N}$ throughout.

Theorem (Belanger–Chong–Wang–W–Yang)

Every countable $M \models \mathrm{I}\Sigma_n + \exp$ has an $(n+1)$-elementary cofinal extension $K \models \mathrm{B}\Sigma_{n+1}(1/2) + \exp$ that does not turn any false instance of $\mathrm{B}\Sigma_{n+1}$ true.

A *proper cut* of M is a nonempty proper subset of M that has no maximum and is closed downwards.

Proof sketch

- Since M is countable, it suffices to show how to build an $(n+1)$-elementary cofinal extension of M that turns *one* arbitrarily chosen instance of $\mathrm{B}\Sigma_{n+1}(1/2) + \exp$ true, while keeping all false instances of $\mathrm{B}\Sigma_{n+1}$ false.
- We build a coded ultrapower K that is designed to make the arbitrarily chosen instance of $\mathrm{B}\Sigma_{n+1}(1/2) + \exp$ true.
- Let $(I_j)_{j \in \mathbb{N}}$ enumerate all Δ_{n+1}-definable proper cuts of M.
- Ensure no $j \in \mathbb{N}$ and no $c \in K$ satisfy $I_j < c < M \setminus I_j$.
- (Yokoyama) Then each I_j has a Δ_{n+1} definition in M that defines a proper cut $\sup_K(I_j)$ of K.
- (Slaman 2004) So all false instances of $\mathrm{B}\Sigma_{n+1}$ in M remain false in K. □

Any* subset of a set that supports a failure of $\mathrm{B}\Sigma_{n+1}(1/2) + \exp$ of large M-cardinality must support the same.

Partial recursive saturation

Fix $n \in \mathbb{N}$ throughout.

Proposition

If $\mathbb{N}$ is Σ_{n+1}-definable in $M \models \mathrm{I}\Sigma_n$, then $\mathbb{N}$ is Σ_{n+1}-definable in all $K \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^-.

Proof sketch

The Σ_{n+1} definability of $\mathbb{N}$ is witnessed by a non-decreasing Σ_{n+1}-definable function $\mathbb{N} \to M$ whose image is cofinal in M. □

Proposition

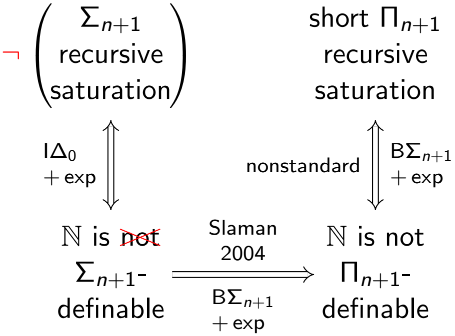
Every countable nonstandard $M \models \mathrm{I}\Sigma_n$ has an $(n+1)$-elementary cofinal extension $K \models \mathrm{B}\Sigma_{n+1}$ in which $\mathbb{N}$ is not Π_{n+1}-definable.

Proof sketch

Iteratively use Compactness to add nonstandard elements to all Π_{n+1}-definable sets that may define $\mathbb{N}$. □

Let $M \models \mathrm{PA}^-$.

- View $\mathbb{N} \subseteq_{\mathrm{e}} M$.
- M is *nonstandard* if $M \neq \mathbb{N}$.
- $c \in M$ is *nonstandard* if $c \notin \mathbb{N}$.



Arithmetic gets better in a cofinal extension

Fix $n \in \mathbb{N}$ throughout.

If $M \models \mathrm{B}\Sigma_{n+1}$, then

- (Gaifman–Dimitracopoulos 1980) all $K \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^- are $(n+2)$-elementary.

If $M \models \mathrm{I}\Sigma_n$, then

- (corollary of the above) all $K \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^- are $(n+1)$-elementary;
- (W) provided $n \geqslant 1$, all $K \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^- also satisfy $\mathrm{B}\Sigma_n$;
- (Paris 1981; Clote–Hájek–Paris 1990) some $K \succcurlyeq_{n+1,\mathrm{cf}} M$ satisfies $\mathrm{B}\Sigma_{n+1}$;
- (Belanger–Chong–Wang–W–Yang) provided M is countable and $M \models \exp + \neg\mathrm{B}\Sigma_{n+1}(1/2)$, some $K \succnsim_{n+1,\mathrm{cf}} M$ does not turn any false instance of $\mathrm{B}\Sigma_{n+1}$ true;
- provided $\mathbb{N} \in \Sigma_{n+1}\text{-Def}(M)$, all $K \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^- can Σ_{n+1}-define $\mathbb{N}$;
- provided M is countable and nonstandard, some $K \succcurlyeq_{n+1,\mathrm{cf}} M$ satisfying $\mathrm{B}\Sigma_{n+1}$ cannot Π_{n+1}-define $\mathbb{N}$.

Question

Can arithmetic (i.e., induction/saturation) get worse in a cofinal extension?